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VOLUME GROWTH AND INTEGRAL CURVATURE BOUND FOR NON-NEGATIVELY CURVED THREE-MANIFOLDS

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Abstract. Motivated by results in Kähler geometry, in this work, we are interested in understanding the relation between integral curvature bounds and volume growth, under non-negative curvature in dimension three. In case of non-negative sectional curvature, we show that for a metric on Euclidean space, it is of Euclidean volume growth if and only if it has average quadratic curvature decay. This is based on showing that metrics on three-dimensional Euclidean space with non-negative sectional curvature are of Euclidean volume growth if their asymptotic scaling invariant integral of scalar curvature is smaller than the sharp constant 8π. We also show a gap theorem if the curvature decays fast enough in the average sense, under non-negative Ricci curvature.

In memory of Prof. Luen-Fai Tam

1. Introduction

Let (M^n, g) be a complete non-compact manifold with non-negative curvature. The celebrated Theorem by Cheeger-Gromoll-Meyer [11, 27] stated that if its sectional curvature $\sec(g)$ is non-negative, then M is diffeomorphic to the normal bundle of a compact totally convex, totally geodesic embedded sub-manifold $\mathcal{S}$, called the soul of M. When $n = 3$, under the weaker assumption $\mathrm{Ric}(g) \geq 0$ it was shown by Liu [31], building on the work of Schoen-Yau [47], that either M^3 is diffeomorphic to $\mathbb{R}^3$ or its universal cover $\tilde{M} = \Sigma^2 \times \mathbb{R}$ splits. The higher dimensional analogy of differentiable structure under curvature conditions weaker than $\sec \geq 0$ is in general unclear. It is also conjectured by Topping [48] that non-negative 1-isotropic curvature is the correct way to generalize $\mathrm{Ric} \geq 0$ for such a topological result.

Nevertheless, the differentiable structure under strong non-negative curvature is generally well-understood. We are interested in the analytic aspect induced from non-negative curvature on non-compact manifolds. The underlying philosophy is that restriction of topological structure should also impose certain analytic constraints, particularly at spatial infinity. When $n = 2$, it was first shown by Cohn–Vossen [13] that under $\sec \geq 0$, we must have $||\mathrm{scal}||_{L^1} \leq 4\pi$. In particular, its total curvature is finite. Later, Huber [24] generalized this result under the assumption that the negative part of the scalar curvature is integrable, and proved that such open surfaces must be conformally equivalent to a compact surface with finitely many points removed. In general dimension $n \geq 3$, it was generalized by Petrunin [45] (see also [30]) that a complete Riemannian manifold (M^n, g) with $\sec \geq 0$ must satisfy

$$\sup_{r>0} \; r^{2-n} \int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g \leq C_n, \tag{1.1}$$

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for all $x \in M$, where C_n is a constant depending only on n. Very recently, it was shown by Hao-Zhu [22] that the same conclusion cannot hold under $\mathrm{Ric} \geq 0$, thereby resolving a problem of Yau [55], see also the independent work by Cheng [12] and Xu [53]. See Yang's work [54] for the counterexample to Yau's conjecture concerning k-th elementary symmetric functions of Ric with $k \geq 2$. Building on the classification theorem [7, 58] for locally conformally flat manifolds with $\mathrm{Ric} \geq 0$, Ma [34] showed (1.1) holds in this special setting. The examples in [12, 22, 53] are volume collapsed at infinity, while the volume non-collapsed case is in general still open and is conjectured by Naber [38]. Specifically, when $\mathrm{Ric}_g \geq 0$ with Euclidean volume growth, whether (1.1) holds. We refer readers to [16, 17] for some progress in the non-collapsed case.

Motivated by these, we are particularly interested in understanding the interplay between volume non-collapsing and integral curvature boundedness, under non-negative curvature. In the Kähler case, it has been studied extensively as part of the program in determining the complex structure of complete non-compact Kähler manifolds with positive bisectional curvature $\mathrm{BK} > 0$. We say BK is quasi-positive if $\mathrm{BK} \geq 0$ and $\mathrm{BK} > 0$ at one point. In particular, the following is true by the joint effort of Ni, Ni-Tam and Liu:

Theorem 1.1 ([39, 42, 32, 33])**.** *Suppose (M^n, g) is a complete non-compact Kähler manifold such that $\dim_{\mathbb{C}}(M) = n \geq 2$ and BK is quasi-positive, then the following are equivalent:*

(i) $\mathrm{AVR}_{\mathbb{C}}(g) := \lim_{r\to+\infty} \dfrac{\mathrm{Vol}_g(x_0, r)}{\omega_{2n} r^{2n}} > 0$;
(ii) *There is non-constant holomorphic function f of polynomial growth;*
(iii) *There is $C > 0$ such that for all $x \in M$ and $r > 0$,*

$$k(x, r) := r^2 \fint_{B_g(x,r)} \mathrm{scal}\, d\mathrm{vol}_g \leq C. \tag{1.2}$$

It naturally raises the question of whether (i) and (iii) in Theorem 1.1 are equivalent in any (purely) Riemannian context. For n-dimensional Riemannian manifold (M^n, g), we define

$$\mathrm{AVR}(g) := \lim_{r\to+\infty} \frac{\mathrm{Vol}_g(x_0, r)}{\omega_n r^n}.$$

The study is particularly active when $n = 3$ under $\mathrm{Ric} \geq 0$. For instance, Xu [52] proved that any complete manifold (M^3, g) with $\mathrm{Ric} \geq 0$ and a pole must satisfy the sharp identity

$$\lim_{r\to+\infty} \frac{1}{r} \int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g = 8\pi(1 - \mathrm{AVR(g)}), \tag{1.3}$$

which generalizes a previous estimate by Zhu [57]. Assuming Euclidean volume growth but not the existence of a pole, a version of (1.3) weighted by the Green function was recently obtained by Chen-Xu-Zhang [10], building on the earlier work of Xu [51]. We also refer interested readers to [37, 56] for results toward generalizing Petrunin's estimate [45], as well as to [46] for a result establishing $\mathrm{AVR}(g) > 0$ under pointwise quadratic curvature decay.

The first theme of this work is to understand how (iii) implies (i), under $\sec \geq 0$. Our first main result is a volume gap theorem of Petrunin's curvature estimate [45]:

Theorem 1.2. *Suppose g is a complete metric on $\mathbb{R}^3$ such that the sectional curvature $\sec(g)$ is non-negative. If there exists $x_0 \in \mathbb{R}^3$ such that*

$$\alpha := \limsup_{r\to+\infty} \frac{1}{r} \int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g < 6\omega_3 := 8\pi$$

then $(\mathbb{R}^3, g)$ has Euclidean volume growth with

$$\mathrm{AVR}(g) \geq 1 - \frac{\alpha}{8\pi}.$$

It is worth mentioning that the constant 8π is sharp even on $\mathbb{R}^3$. The rotationally symmetric metric with an asymptotically cylindrical end,

$$g := dr^2 + \tanh^2(r) g_{\mathbb{S}^2} \tag{1.4}$$

has $\sec \geq 0$ and $\mathrm{AVR}(g) = 0$, whereas $\alpha = 8\pi$. This in particular gives a three-dimensional analogue of Theorem 1.1 under $\sec \geq 0$:

Corollary 1.3. *Suppose (M^3, g) is a complete non-compact non-flat manifold such that* $\sec \geq 0$ *and* $\mathrm{Ric}(x_0) > 0$ *for some $x_0 \in M$, then the following are equivalent:*

(i) $\mathrm{AVR}(g) > 0$;

(ii) $\limsup_{r\to+\infty} r^2 \fint_{B_g(x,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g < +\infty$, *for all $x \in M$.*

Theorem 1.2 is in other words saying that the curvature of a volume collapsed manifold $(\mathbb{R}^3, g)$ with $\sec \geq 0$ must be at least 8π in integral sense asymptotically, i.e. $\alpha \geq 8\pi$. Although the geometric quantity α detects non-flatness at spatial infinity, it is at the same time influenced by local geometry. More generally, we show that the integral curvature grows linearly. The same also holds in general dimension $n \geq 3$:

Theorem 1.4. *Suppose (M^n, g) is a complete non-compact manifold such that* $\sec(g) \geq 0$ *and* $\mathrm{rank}(\mathrm{Ric}(x_0)) \geq 3$ *for some $x_0 \in M$, then there exists $c_1 > 0$ such that for all $r > 1$,*

$$\frac{1}{r} \int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g \geq c_1.$$

In particular, the total scalar curvature is infinite.

In fact, the constant c_1 in Theorem 1.4 can be explicitly controlled by curvature positivity at x_0, see Proposition 5.5. Moreover, the linear growth is sharp under $\sec > 0$. This can be seen from the example above, i.e. the rotationally symmetric metric g in (1.4) with $\mathbb{S}^2$ replaced by $\mathbb{S}^{n-1}$ for general $n \geq 3$.

The second theme of this work is to explore the gap phenomenon of three-manifolds with $\mathrm{Ric} \geq 0$ and vanishing asymptotic curvature, i.e. $k(x_0, r)$ in (1.2) is $o(1)$ as $r \to +\infty$. Despite the curvature positivity, this is the extreme case appearing in Corollary 1.3 where the geometric quantity in (ii) vanishes. Gap theorems of this kind in the Riemannian context were studied by the first two named authors [9]. The purpose there is to develop a Riemannian analogue of the optimal gap Theorem of Ni [41] (see also [43]) asserting that a complete non-compact Kähler manifold (M^n, g) with $\mathrm{BK} \geq 0$ is flat if $k(x_0, r) = 0$ as $r \to \infty$ for some $x_0 \in M$. Our motivation is to explore the Riemannian counterpart of the Theorem of Ni [41], with the weakest possible curvature condition. In [9], the first two named authors used the Ricci flow approach to answer it in the non-collapsed case under non-negative 1-isotropic curvature assumption[1]. And thus the remaining challenge is to treat the collapsing situation. When $n = 3$, we exploit the potential theory of p-harmonic functions and prove the optimal gap Theorem:

Theorem 1.5. *Suppose (M^3, g) is a complete non-compact manifold such that* $\mathrm{Ric} \geq 0$ *and there exists $x_0 \in M$ such that*

$$\limsup_{r\to+\infty} r^2 \fint_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g = 0,$$

then (M^3, g) is flat.

[1]The combined methodologies in [9, 8] also implies the gap Theorem for $n = 3$ in the non-collapsed case.

Most results of this work rely on three-dimensionality for the purpose of applying the Gauss-Bonnet Theorem on various kinds of level sets. In the first part, we study (M^n, g) with $\sec \geq 0$ by exploiting the Busemann function $b : M \to \mathbb{R}$ and its properties from [11]. It is a 1-Lipschitz function which is proper and convex, if $\sec \geq 0$. To address the volume growth in Theorem 1.2, we consider the sub-level set C_ℓ of the Busemann function b. Our topological assumption $M^3 = \mathbb{R}^3$ ensures that the soul $\mathcal{S}$ of M is a point so that C_ℓ are contractible. A very classical work [5] of Bujalo, based on Gauss-Bonnet, asserts that the total scalar curvature on C_ℓ and the total (generalized) mean curvature on ∂C_ℓ can be controlled by the model on $\mathbb{R}^3$ sharply. This in turn controls the volume growth of M by the variational formula of $\mathrm{Vol}_g(C_\ell)$ and distance comparison, modulo the error from total curvature decay α.

When we relax $\sec \geq 0$ to $\mathrm{Ric} \geq 0$, the Busemann function b is only subharmonic in general. Moreover, when $n \geq 4$, it is in general not proper by the examples [44] of Pan-Wei, while the case $n = 3$ is still open. We consider an alternative choice of exhaustion using Green type function. Under $\mathrm{Ric} \geq 0$, the properness of Green-type function is tied to its volume growth. In the setting of Theorem 1.5, assume on the contrary (M^3, g) is non-flat, the volume growth must be super-quadratic, thanks to $\lim_{r\to+\infty} k(x_0, r) = o(1)$. For some well-chosen $p \in (1, 3]$, this in particular implies (M^3, g) is p-nonparabolic and its minimal p-Green function is proper. In reality, we consider instead p-capacitary function u on some exterior domain $M \setminus B_g(x_0, \delta)$ and use its level set to exhaust $M \setminus B_g(x_0, \delta)$. Exploring the monotonicity developed by [2] with the help of gradient estimate of p-harmonic type functions, we are able to show that M must be of quadratic volume growth, contradicting super-quadratic in the first place. The circle of ideas also extends to a discussion of volume growth, in a similar spirit as Theorem 1.2. For a more general statement, we refer readers to Theorem 6.1.

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2. Properties of Busemann function

In this section, we collect some preliminaries about the Busemann function on manifolds with $\sec \geq 0$ which was studied extensively by Cheeger-Gromoll-Meyer [11, 27]. Fix $x_0 \in M$ and denote $\rho(x) = d_g(x_0, x)$. We consider the set $\mathcal{R}$ consisting of all rays $\gamma : [0, +\infty) \to M$ with $\gamma(0) = x_0$. The Busemann function $b : M \to \mathbb{R}$ is defined to be

$$b(x) := \sup_{\gamma\in\mathcal{R}} \left\{ \lim_{s\to+\infty} (s - d_g(\gamma(s), x)) \right\}.$$

By the celebrated work of Cheeger-Gromoll-Meyer [11, 27], it is known that b is a convex function on M and thus $\inf_M b < +\infty$. By translation, we will assume $\inf_M b = 0$. We consider the exhaustion of M by its sub-level sets. For all $\ell > 0$, denote

$$C_\ell := \{x \in M : b(x) \leq \ell\} \subseteq M.$$

The following properties will be important for us.

Lemma 2.1. *For all $\ell > 0$, we have*

(i) *Each C_ℓ is a totally convex compact set.*
(ii) $\dim(C_\ell) = n$ *and* $\cup_{\ell>0} C_\ell = M$.
(iii) C_ℓ *has the structure of an embedded sub-manifold of M with smooth totally geodesic interior and (possibly non-smooth) boundary.*

Proof. It follows from [11, Proposition 1.3 & Theorem 1.6]. □

We also need the relation between $b(x)$ and $\rho(x)$.

Lemma 2.2. *Under* $\sec \geq 0$*, the Busemann function b satisfies the following:*
(i) *b is a 1-Lipschitz function on M;*
(ii) *There exists a function $\theta(s)$ with $\lim_{s\to+\infty} \theta(s) = 0$ such that on M we have*

$$(1 - \theta \circ \rho)\rho \leq b \leq \rho.$$

Proof. The property (i) follows from definition and triangle inequality. The property (ii) follows from Toponogov's comparison theorem, see [18, Lemma 1]. □

In particular, Lemma 2.2 provides us with the lower bound for in-radius of each C_ℓ.

Lemma 2.3. *The in-radius of C_ℓ, defined by*

$$t_0(C_\ell) := \sup\{r > 0 : B(x, r) \Subset C_\ell\},$$

satisfies $t_0(C_\ell) \geq \ell$.

Proof. By Lemma 2.2, $B(x_0, \ell) \subseteq C_\ell$ and hence $\ell \leq t_0(C_\ell)$. □

Lemma 2.4. *The Busemann function b is differentiable a.e. and satisfies $|\nabla b| = 1$ at points where it is differentiable. Furthermore,*

$$\mathrm{Vol}_g(C_\ell) = \int_0^\ell A(t)\, dt$$

where $A(t) := \mathcal{H}^{n-1}(\partial C_t)$ for $t > 0$.

Proof. By Lemma 2.2 and Rademacher's theorem, the Busemann function b is differentiable almost everywhere. At x where b is differentiable, we have $|\nabla b| \leq 1$, since b is 1-Lipschitz. It remains to prove the lower bound. We denote $b_\gamma(x) := \lim_{s\to+\infty} (s - d_g(\gamma(s), x))$. We choose a sequence of rays $\{\gamma_j\}$ such that

$$b_{\gamma_j}(x) \geq b(x) - j^{-1}. \tag{2.1}$$

For each γ_j, take $t_i \to +\infty$ and consider the minimizing geodesic $\sigma_{i,j}(s)$ from x to $\gamma_j(t_i)$. By compactness of geodesic, $\sigma_{i,j}$ sub-converges to geodesic σ_j. On the other hand for all $s \in [0, d_g(x, \gamma_j(t_i)]$,

$$\begin{aligned}\lim_{i\to+\infty} (t_i - d_g(\sigma_{i,j}(s), \gamma_j(t_i))) &= s + \lim_{i\to+\infty} (t_i - d_g(\gamma_j(t_i), x)) \\ &= s + b_{\gamma_j}(x)\end{aligned}$$

while the left hand side equals to $b_{\gamma_j}(\sigma_j(s))$. Therefore,

$$b_{\gamma_j}(x) = b_{\gamma_j}(\sigma_j(s)) - s$$

for all $s \geq 0$. We also assume σ_j sub-converges to geodesic σ as $j \to +\infty$. This implies

$$\begin{aligned} b(\sigma(s)) &= \lim_{j\to+\infty} b(\sigma_j(s)) \\ &\geq \lim_{j\to+\infty} b_{\gamma_j}(\sigma_j(s)) \\ &= \lim_{j\to+\infty} b_{\gamma_j}(x) + s \geq b(x) + s. \end{aligned}$$

where we have used (2.1). On the other hand since b is 1-Lipschitz,

$$b(\sigma(s)) - b(x) \leq d_g(\sigma(s), x) = d_g(\sigma(s), \sigma(0)) = s.$$

This shows that $b(\sigma(s)) = b(x) + s$. If we let $v := \sigma'(0)$, then

$$\nabla b|_x(v) = \frac{d}{ds}\Big|_{s=0} b(\sigma(s)) = 1.$$

This shows that $|\nabla b|(x) = 1$.

We now prove the second assertion. By co-area formula and $|\nabla b| = 1$ a.e.,

$$\begin{aligned}\mathrm{Vol}_g(C_\ell) &= \int_{C_\ell} d\mathrm{vol}_g \\ &= \int_{C_\ell} |\nabla b|\, d\mathrm{vol}_g = \int_0^\ell A(t)\, dt.\end{aligned}$$

□

Lemma 2.5. *Suppose the soul is a point and $n = 3$, then for all $\ell > 0$, we have*

$$\chi(\partial C_\ell) = 2.$$

Proof. By [11, Theorems 2.1 and Theorem 2.5], C_s is homotopy equivalence to each other. Since the soul is a point, it follows that C_ℓ is contractible and thus the result follows. □

3. Bujalo's convex body inequality in dimension three

In this section, we discuss the result of Bujalo [5] on an inequality in controlling the integral of curvature by topological information. We first collect some materials used by Bujalo [5]. We will keep the discussion minimal and refer readers to [5] for the detailed exposition. We follow the discussion of [5, Section 2]. We assume $\sec \geq 0$ and $n = 3$ throughout this section.

For each closed locally convex set with nonempty interior $C \Subset M^3$, by the work [49] of Walter there is an open set U containing C such that:

(i) For each $q \in U$, there is a unique $\Phi(q) \in C$, and a unique minimal geodesic from q to $\Phi(q)$ which lies entirely in U.
(ii) The function $f(q) := d_g(q, C)$ for $q \in U$, belongs to $C^{1,1}$.

In particular, $|\nabla f| = 1$ on $U \setminus C$ and the map Φ is given by

$$\Phi(q) = \exp_q\left(-f(q)\nabla f(q)\right).$$

We consider $C_\tau := \{q \in M : d_g(q, C) \leq \tau\}$ and the equidistance surface $F_\tau := \{q \in M : d(q, C) = \tau\}$ where F_τ is a $C^{1,1}$-smooth hypersurface for all small $\tau > 0$. We fix $N := F_{\tau_0}$ for sufficiently small $\tau_0 > 0$ and consider the map $\varphi_\tau := \Phi_\tau|_N : N \to F_\tau$ for $\tau \in (0, \tau_0]$, where

$$\Phi_\tau(q) := \exp_q\left((\tau - f(q))\nabla f(q)\right).$$

For $\tau \in (0, \tau_0]$, the map φ_τ is Lipschitz and hence differentiable a.e. on N by Rademacher's theorem. The Hausdorff measure of ∂C is thus given by

$$\mathcal{H}^2(\partial C) = \int_N I_0(p)\, dA_N,$$

by [5, Lemma 2.3.1], see also [19, Theorem 3.2.3]. Furthermore, this can be recovered by approximation according to [5, Lemma 2.3.2]:

$$\mathcal{H}^2(\partial C) = \lim_{\tau \to 0^+} \int_N I_\tau(p)\, dA_N$$

Here I_τ denotes the Jacobian of φ_τ which exists a.e. on N.

With this terminology, the mean curvature of ∂C is defined through that of F_τ. More precisely, there exists $N' \subset N$ with $\mathcal{H}^{n-1}(N \setminus N') = 0$ such that the second fundamental form h_τ is defined at $\varphi_\tau \in F_\tau$ for all $\tau \in (0, \tau_0]$. Moreover, the mean curvature $\mathbf{H}(\tau, p) \cdot I_\tau(p)$ (weighted with its Jacobian) is uniformly bounded with finite limit

$$\mathbf{H}(p) := \lim_{\tau \to 0^+} \mathbf{H}(\tau, p) \cdot I_\tau(p) \geq 0,$$

from [5, Lemma 2.4.6 & Lemma 2.4.7]. From [5, Corollary 2.4.8], the total mean curvature $\mathcal{M}(\partial C)$ of ∂C is then defined to be

$$\mathcal{M}(\partial C) := \int_N \mathbf{H}(p)\, dA_N. \tag{3.1}$$

We are now ready to state the main inequality in this section.

Proposition 3.1. *For all $s > 0$, we have*

$$4\pi\chi(\partial C_s) \cdot t_0(C_s) \leq \mathcal{M}(\partial C_s) + \int_{C_s} \text{scal}\ d\text{vol}$$

where $t_0(C)$ denotes the in-radius of a set C, defined in Lemma 2.3.

Proof. Since C_s is a closed convex set by Lemma 2.1, this follows from applying [5, Theorem 5.2.1] to the sublevel set C_s of the Busemann function. □

Remark 3.2. If C_s in Proposition 3.1 has a smooth boundary ∂C_s, then $\mathcal{M}(\partial C_s)$ coincides with the total mean curvature of ∂C_s by construction.

3.1. **Minkowski inequality on convex body.** In this section, we will establish a radial Minkowski inequality relating the generalized total mean curvature of the boundary of the level set C_s of the Busemann function and its Hausdorff measure.

Theorem 3.3. *For every $s > 0$, we have*

$$s \cdot \mathcal{M}(\partial C_s) \leq 2\mathcal{H}^2(\partial C_s).$$

To make the idea more transparent, we split it into the smooth case and the general case. The general case requires the terminology introduced in Section 3.

3.1.1. *Smooth case.* We first discuss the proof of Theorem 3.3 when all objects are smooth.

Proposition 3.4. *Theorem 3.3 holds when ∂C_s is a smooth hypersurface.*

Proof. Let ν be the outward unit normal to ∂C_s. In our convention, the second fundamental form is given by $h(X, Y) = \langle \nabla_X \nu, Y \rangle$. We consider the function $u := \frac{1}{2}\rho^2$. By Hessian comparison and $\sec \geq 0$, we have

$$\nabla^2 u \leq g$$

in the sense of support functions and hence its restriction on $\Sigma = \partial C_s$, u satisfies

$$\Delta_\Sigma u = \text{tr}_{T\Sigma} \nabla^2 u - H \cdot \partial_\nu^- u \leq 2 - H \cdot \partial_\nu^- u, \tag{3.2}$$

in the sense of a barrier, and thus a distribution [35, Appendix A]. Here $\partial_\nu^- u$ is the least one-sided normal derivative determined by the minimizing geodesic from p.

By integrating (3.2) over Σ, we conclude

$$\int_\Sigma H \cdot \partial_\nu^- u\ d\text{A}_\Sigma \leq 2 \cdot \text{Area}(\Sigma). \tag{3.3}$$

Since $H \geq 0$, it suffices to control $\partial_\nu u$ from below. Fix $x \in \Sigma$, we let $\gamma : [0, L] \to M$ be a minimizing geodesic from x_0 to x with length L. Then

$$\begin{aligned}\partial_\nu^- u &= \langle \nu, \nabla u \rangle = \rho \langle \nabla b, \nabla \rho \rangle \\ &= L \cdot \langle \nabla b, \gamma'(L) \rangle \\ &= L \cdot \frac{d}{dt}\Big|_{t=L^-} (b \circ \gamma)(t).\end{aligned}$$

If we denote $b \circ \gamma$ by f, then the convexity of b implies f is convex. Since $f(0) = 0$ and $f(L) = s$, this implies $f'(L) \geq sL^{-1}$ and therefore,

$$\partial_\nu^- u = L \cdot \frac{d}{dt}\Big|_{t=L^-} f(t) \geq s. \tag{3.4}$$

By substituting (3.4) into (3.3), this implies

$$s \cdot \int_{\partial C_s} H \, d\mathrm{A}_{\partial C_s} \leq 2 \cdot \mathrm{Area}(\partial C_s).$$

This completes the proof. □

3.1.2. *General case.* We now prove Theorem 3.3 in full generality, i.e. when $\Sigma := \partial C_s$ is still convex but is possibly non-smooth.

Proof of Theorem 3.3. Following the discussion in section 3, we consider

$$C_{s,\tau} := \{q \in M : d_g(q, C_s) \leq \tau\}, \text{ and } \Sigma_\tau := \{q \in M : d_g(q, C_s) = \tau\}.$$

We also denote $N := \Sigma_{\tau_0}$ for sufficiently small $\tau_0 > 0$. For all $\tau \in (0, \tau_0]$, each Σ_τ is a closed $C^{1,1}$ hypersurface in M. We let N' be the set where $\mathcal{H}^2(N \setminus N') = 0$ where $h_\tau \circ \varphi_\tau$ is defined on N'. By [5, Lemma 2.2.18 & (2.2.2)], its second fundamental form $h_\tau \circ \varphi_\tau(p)$ satisfies

$$-C_0 \tau \leq h_\tau \circ \varphi_\tau \leq C_1 \tau^{-1}.$$

By [6, Lemma 5.6], we might further regularize each $C^{1,1}$ hypersurface Σ_τ by smooth hypersurfaces $\Sigma_{\tau,j}$ such that $\Sigma_{\tau,j}$ converges to Σ_τ in C^1 topology and satisfies

$$-C_0 \tau - j^{-1} \leq h_{\Sigma_{\tau,j}} \leq C_1 \tau^{-1} + j^{-1}. \tag{3.5}$$

Since $\Sigma_{\tau,j}$ is smooth, we might apply (3.3) from the proof of Proposition 3.4 to deduce

$$\int_{\Sigma_{\tau,j}} H_{\Sigma_{\tau,j}} \cdot \partial_\nu^- u \, d\mathrm{A}_{\Sigma_{\tau,j}} \leq 2 \cdot \mathrm{Area}(\Sigma_{\tau,j}).$$

for all $\tau \in (0, \tau_0]$ and $j \to +\infty$. Using (3.5) and $|\nabla u| \leq C_1$,

$$\inf_{\Sigma_{\tau,j}} \partial_\nu^- u \cdot \int_{\Sigma_{\tau,j}} H_{\Sigma_{\tau,j}} d\mathrm{A}_{\Sigma_{\tau,j}} \leq \big(2 + C_2(\tau + j^{-1})\big) \cdot \mathrm{Area}(\Sigma_{\tau,j})$$

By the two-sided bound of the second fundamental form (3.5), we might pass $j \to +\infty$ to conclude

$$\inf_{\Sigma_\tau} \partial_\nu^- u \cdot \int_N \mathbf{H}(\tau, p) \cdot I_\tau(p) d\mathrm{A}_N \leq (2 + C_2 \tau) \cdot \mathrm{Area}(\Sigma_\tau) \tag{3.6}$$

where $\inf_{\Sigma_\tau} \partial_\nu^- u$ denotes the infimum of least one-sided normal derivatives determined by the minimizing geodesic from x_0 to $x \in \Sigma_\tau$.

We next claim that the normal derivative $\partial_\nu^- u$ is asymptotically bounded from below by s as in the smooth case.

Claim 3.1. As $\tau \to 0^+$, we have

$$\liminf_{\tau \to 0^+} \left(\inf_{\Sigma_\tau} \partial_\nu^- u \right) \geq s$$

Proof of Claim 3.1. Suppose on the contrary, there is $\delta > 0$, $\tau_i \to 0^+$, $x_i \in \Sigma_{\tau_i}$, outer normal ν_i and a minimizing geodesic from x_0 to x_i whose terminal unit vector v_i satisfies

$$d_g(x_0, x_i) \cdot \langle \nu_i, v_i \rangle \leq s - \delta.$$

Denote $y_i = \Phi(x_i)$ where Φ is the nearest-point projection onto C_s. Then compactness implies $x_i, y_i \to x_\infty \in \partial C_s$, $\nu_i \to \nu_\infty$ and $v_i \to v_\infty$ where ν is an outward unit support normal to C_s at x_∞ and v_∞ is the terminal direction of a minimizing geodesic γ_∞ from $x_0 = \gamma_\infty(0)$ to $x_\infty = \gamma_\infty(L)$ so that

$$L \cdot \langle \nu_\infty, \gamma_\infty'(L) \rangle \leq s - \delta. \tag{3.7}$$

At x_∞, for some $\lambda \in (0, 1]$ $\lambda\nu_\infty$ is a sub-gradient and thus $\lambda\langle \nu_\infty, \gamma_\infty'(L)\rangle \geq sL^{-1}$ which contradicts (3.7). This proves the claim. □

By Claim 3.1, (3.6) and (3.1), we might let $\tau \to 0$ to conclude Theorem 3.3. □

4. Proof of Theorem 1.2

In this section, we show that a universal bound 8π of the average integral curvature

$$\hat{k}(x_0, r) := r^{-1} \int_{B_g(x_0,r)} \mathrm{scal}\, d\mathrm{vol}_g$$

will force M^3 to be non-collapsed at infinity. We isolate one direction of inequality in case of maximal volume growth, which holds only under $\mathrm{Ric} \geq 0$.

Proposition 4.1. *Suppose (M^3, g) is a complete non-compact manifold such that* $\mathrm{Ric} \geq 0$ *and* $\mathrm{AVR}(g) > 0$*, then for any $x_0 \in M$,*

$$\liminf_{r\to+\infty} \frac{1}{r} \int_{B_g(x_0,r)} \mathrm{scal}\, d\mathrm{vol}_g \geq 8\pi\,(1 - \mathrm{AVR}(g)) \geq 0.$$

Proof. Let $u := (4\pi G_{x_0})^{-1}$ and G_{x_0} is the minimal positive Green function with pole x_0 satisfying $-\Delta G_{x_0} = \delta_{x_0}$, then [10, Theorem 1.5] gives us

$$\begin{cases} \lim_{t\to+\infty} t^{-1} J(t) = 8\pi(1 - \mathrm{AVR}(g)); \\ J(t) := \displaystyle\int_{\{u\leq t\}} \mathrm{scal} \cdot |\nabla u|\, d\mathrm{vol}_g. \end{cases} \tag{4.1}$$

On the other hand, [14, Theorem 3.5] implies

$$\lim_{r\to+\infty} \sup_{M\setminus B_g(x_0,r)} |\nabla u| = \mathrm{AVR}(g).$$

Together with [15, (3.37)], for $\varepsilon > 0$, we choose $s_\varepsilon > 0$ such that outside $B_g(x_0, s_\varepsilon)$,

$$u(x) \geq (\mathrm{AVR} - \varepsilon) d_g(x, x_0), \;\; |\nabla u| \leq \mathrm{AVR} + \varepsilon.$$

Then $\{u < (\mathrm{AVR} - \varepsilon)r\} \subset B_g(x_0, r)$ for all large r and hence,

$$r^{-1} J\left((\mathrm{AVR} - \varepsilon)r\right) \leq C_\varepsilon r^{-1} + (\mathrm{AVR} + \varepsilon) r^{-1} \int_{B_g(x_0,r)} \mathrm{scal}\, d\mathrm{vol}_g. \tag{4.2}$$

Result follows by letting $r \to +\infty$ on (4.2) using also (4.1) and followed by $\varepsilon \to 0$. □

Proof of Theorem 1.2. By the Soul Theorem, the soul $\mathcal{S}$ is a point as the manifold is topologically Euclidean. We fix $x_0 \in M$ and consider the Busemann function b on M.

By Lemma 2.2, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that $(1+\varepsilon)^{-1}\rho \leq b$ for all $\rho > R_\varepsilon$. In particular, for all s large, $C_s \subseteq B_g(x_0, (1+\varepsilon)s)$, and hence

$$\int_0^s A(t)\, dt \leq \mathrm{Vol}_g(C_s) \leq \mathrm{Vol}_g\left(x_0, (1+\varepsilon)s\right) \tag{4.3}$$

by Lemma 2.4, where $A(t) := \mathcal{H}^2(\partial C_t)$. We also assume R_ε is large enough such that

$$\frac{1}{r}\int_{B_g(x_0,r)} \operatorname{scal} d\mathrm{vol}_g < \alpha + \varepsilon < 8\pi \tag{4.4}$$

for all $r > R_\varepsilon$.

By Lemma 2.3, Lemma 2.5 and Proposition 3.1, we have

$$\begin{aligned} 8\pi t &\le 4\pi\chi(\partial C_t)\cdot t_0(C_t) \\ &\le \mathcal{M}(\partial C_t) + \int_{C_t} \operatorname{scal} d\mathrm{vol}_g \end{aligned}$$

Hence Theorem 3.3 implies

$$4\pi t^2 - \frac{1}{2}t\int_{C_t} \operatorname{scal} d\mathrm{vol}_g \le \frac{1}{2}t\cdot\mathcal{M}(\partial C_t) \le A(t) \tag{4.5}$$

Combining (4.5) with (4.3), we conclude

$$\omega_3 s^3 \le \mathrm{Vol}_g\,(x_0,(1+\varepsilon)s) + \frac{1}{2}\int_0^s\int_{C_t} t\cdot\operatorname{scal} d\mathrm{vol}_g \tag{4.6}$$

for all s large.

For the last integral, we split it into

$$\begin{aligned} \frac{1}{2}\int_0^s\int_{C_t} t\cdot\operatorname{scal} d\mathrm{vol}_g &= \frac{1}{2}\left(\int_{R_\varepsilon}^s + \int_0^{R_\varepsilon}\right)\int_{C_t} t\cdot\operatorname{scal} d\mathrm{vol}_g \\ &=: \mathbf{I} + \mathbf{II} \end{aligned}$$

where $\mathbf{II} = o(s^3)$ for each fixed $\varepsilon > 0$. For $\mathbf{I}$, we use (4.4) so that

$$\begin{aligned} \mathbf{I} &\le \frac{1}{2}\int_{R_\varepsilon}^s t\int_{B_g(x_0,(1+\varepsilon)t)} \operatorname{scal} d\mathrm{vol}_g \\ &\le \frac{1}{2}(\alpha+\varepsilon)(1+\varepsilon)\int_{R_\varepsilon}^s t^2\,dt \le \frac{1}{6}(\alpha+\varepsilon)(1+\varepsilon)s^3. \end{aligned}$$

And hence (4.6) is reduced to

$$1 \le \frac{\mathrm{Vol}_g\,(x_0,(1+\varepsilon)s)}{\omega_3 s^3} + \frac{1}{6\omega_3}(\alpha+\varepsilon)(1+\varepsilon) + o(1)$$

for all $s \to +\infty$. By volume comparison, we pass $s \to +\infty$ and followed by $\varepsilon \to 0$ to show that

$$1 - \frac{\alpha}{6\omega_3} \le \mathrm{AVR}(g) := \lim_{s\to+\infty}\frac{\mathrm{Vol}_g\,(x_0,s)}{\omega_3 s^3}.$$

□

The dichotomy analogy of Theorem 1.1 is now immediate.

Proof of Corollary 1.3. If (i) is true, then (ii) follows directly from Petrunin [45]. The difficult part is to prove the opposite direction. By the work of Liu [31], M^3 is diffeomorphic to $\mathbb{R}^3$. Suppose $\mathrm{AVR}(g) = 0$, then Bishop-Gromov volume comparison yields

$$\frac{1}{r}\int_{B_g(x_0,r)} \operatorname{scal} d\mathrm{vol}_g \le \frac{\mathrm{Vol}_g(x_0,r)}{r^3}\cdot r^2 \fint_{B_g(x_0,r)} \operatorname{scal} d\mathrm{vol}_g.$$

By letting $r \to +\infty$, this shows that $\alpha = 0$ in Theorem 1.2 and hence $\mathrm{AVR} > 0$. This is impossible. □

5. Curvature growth of non-flat manifold

In Theorem 1.2, we see how an integral curvature bound gives rise to non-collapsing. We now show that if in addition $\mathrm{Ric}(x_0) > 0$ somewhere, then $\hat{k}(x_0, r)$ grows at least linearly. Indeed, we will prove the estimates for general $n \geq 3$.

Lemma 5.1. *Suppose* $\sec \geq 0$*, then the tensor*

$$T := \frac{1}{2}\mathrm{scal} \cdot g - \mathrm{Ric}$$

is non-negative and is divergence-free.

Proof. This follows from [9, Lemma 3.2] and Bianchi identity. □

To ease the regularity issue caused in differentiating the Busemann function b, we use the approximation method of Greene-Wu [26]:

Lemma 5.2. *There exists* $\varepsilon_i \to 0^+$ *and a sequence of smooth functions* $b_i \in C^\infty(M)$ *such that*

(1) $|b_i - b| \leq \varepsilon_i$;
(2) $|\nabla b_i| \leq 1$;
(3) $\nabla^2 b_i \geq -\varepsilon_i g$ *on* M.

Proof. It follows from the convexity of b, [26, Proposition 2.3] and Lemma 2.2. □

Now we use b_i to approximate $b_\infty := b$ and consider a signed measure on M as follows. We let $\phi(s) := s + \sqrt{1+s^2}$, $f_i := \phi(b_i)$ and $X_i := T^\sharp \nabla f_i$.

Lemma 5.3. *The function* $\mathrm{div}(X_i)$ *satisfies*

$$\mathrm{div}(X_i) \geq -o_{loc}(1), \ \textit{as} \ i \to +\infty.$$

Proof. By direct computation and Lemma 5.1,

$$\begin{aligned} \mathrm{div}(X_i) &= \langle T, \nabla^2 f_i \rangle \\ &= \phi'(b_i) \cdot \langle T, \nabla^2 b_i \rangle + \phi''(b_i) \cdot T(\nabla b_i, \nabla b_i) \\ &\geq -\varepsilon_i \left(\frac{n}{2} - 1\right) \cdot \phi(b_i) \cdot \mathrm{scal} + \phi''(b_i) \cdot T(\nabla b_i, \nabla b_i) \\ &\geq -\varepsilon_i \left(\frac{n}{2} - 1\right) \cdot \phi(b_i) \cdot \mathrm{scal}, \end{aligned} \tag{5.1}$$

where we have used Lemma 5.2, Lemma 5.1 and $\mathrm{tr}_g T = (\frac{n}{2} - 1)\mathrm{scal}$. This completes the proof by Lemma 5.2. □

Proposition 5.4. *Suppose* (M, g) *is a complete non-compact manifold such that* $\sec \geq 0$ *and* $\mathrm{rank}(\mathrm{Ric}(x_0)) \geq 3$ *for some* $x_0 \in M$*, then for any* $r > 0$ *we have*

$$\liminf_{i \to +\infty} \int_{B_g(x_0, r)} \mathrm{div}(X_i) \, d\mathrm{vol}_g > 0.$$

Proof. Suppose the conclusion fails, by Lemma 5.3 we might assume

$$\lim_{i \to +\infty} \int_{B_g(x_0, r)} \mathrm{div}(X_i) \, d\mathrm{vol}_g = 0, \tag{5.2}$$

after passing to a subsequence.

By (5.1), Lemma 5.1 and Lemma 5.2,

$$\lim_{i \to +\infty} \int_{B_g(x_0, r)} T(\nabla b_i, \nabla b_i) \, d\mathrm{vol}_g = 0. \tag{5.3}$$

By Lemma 2.4, we let $U \subseteq B_g(x_0, r)$ be a subset such that $b = b_\infty$ is differentiable on U and $\mathrm{Vol}_g\left(B_g(x_0, r) \setminus U\right) = 0$. Using (5.3), Lemma 5.1, full measure of U and semi-continuity of rank(Ric), we choose $x_1 \in U$ so that

(a) $\lim_{i\to+\infty} T(\nabla b_i, \nabla b_i)(x_1) = 0$;
(b) $\mathrm{rank}(\mathrm{Ric}(x_1)) \geq 3$.

At x_1, $\nabla b_i \in T_{x_1}M$ such that $|\nabla b_i|(x_1) \leq 1$ by Lemma 5.2. By passing to subsequence, we might assume $\nabla b_i(x_1) \to Y$ as $i \to +\infty$, for some vector $Y \in T_{x_1}M$. In particular, either $Y = 0$ or $Y \in \mathrm{Ker}(T(x_1))$. On the other hand, using the almost convexity of b_i from Lemma 5.2 and C^0 convergence of b_i to b_∞, we conclude that for any geodesic γ starting from $\gamma(0) = x_1$, we have

$$b(\gamma(t)) \geq \langle Y, \gamma'(0)\rangle \cdot t + b(x_1).$$

In particular, Y is a sub-gradient of b at x_1. Since we have assumed b is differentiable at x_1, $Y = \nabla b(x_1)$ and hence $Y \neq 0$ since $|\nabla b|(x_1) = 1$ by Lemma 2.4.

Claim 5.1. We have $\mathrm{Null}(T(x_1)) \leq 1$.

Proof of Claim 5.1. Suppose on the contrary, there is an orthonormal frame $\{e_i\}_{i=1}^n$ in $T_{x_1}M$ such that $T(e_1) = T(e_2) = 0$. Then we have $R_{11} = R_{22} = \frac{1}{2}\mathrm{scal}$ and hence $\mathrm{rank}(\mathrm{Ric}) = 2$ which is impossible. □

By Claim 5.1, we might choose a smooth vector field V on a neighborhood W of x_1 such that $\mathrm{Ker}(T) = \mathrm{span}\{V\}$ on W and $|V| = 1$. We also choose V so that $V(x_1) = Y = \nabla b(x_1) \in T_{x_1}M$.

Claim 5.2. There exists an open neighborhood W' of x_1 such that

$$\nabla b(x) = V(x)$$

on $W' \cap U$. In particular, b is smooth on W'.

Proof of Claim 5.2. Suppose, on the contrary, that there exists a sequence $\{z_k\}_{k=1}^\infty$ such that $z_k \in U$, $z_k \to x_1$ and $\nabla b(z_k) \neq V(z_k)$ for all k. By repeating the above argument to z_k in place of x_1, we must have $\nabla b(z_k) = -V(z_k)$ such that $\nabla b(z_k) \to -V(x_1)$ by continuity of V. But this contradicts the continuity of the subgradient at a convergent differentiable point. This shows that $\nabla b = V$ almost everywhere on some open neighborhood W'. Since b is locally Lipschitz with smooth weak derivatives, b is smooth there. □

Claim 5.2 also shows $\mathrm{div}(X_\infty)$ exists smoothly on W'. By (5.1) and smoothness of b, we want to rule out the situation that on W',

(1) $T(\nabla b) \equiv 0$ and
(2) $\langle T, \nabla^2 b\rangle \equiv 0$.

Since ∇b is non-trivial from Lemma 2.4, T must be of rank $n-1$ on W' using Claim 5.1. From $\langle T, \nabla^2 b\rangle = 0$, $\nabla^2 b = \lambda V \otimes V$ on W' where $\lambda(x) \geq 0$. We now differentiate $|\nabla b|^2 = 1$ to see that $\langle \nabla^2 b, \nabla b\rangle = 0$ and thus ∇b is a parallel vector field on W'.

By the Bochner formula, we therefore conclude that $\mathrm{Ric}(\nabla b, \nabla b) = 0$, while

$$0 = \mathrm{Ric}(\nabla b, \nabla b) = \frac{1}{2}\mathrm{scal} \cdot |\nabla b|^2 > 0,$$

since $\mathrm{rank}(\mathrm{Ric}) \geq 1$ and $\mathrm{Ric} \geq 0$ on W'. This is impossible. In conclusion, we have shown that

$$\int_{W'} \mathrm{div}(X_\infty)\, d\mathrm{vol}_g > 0.$$

Finally we claim that this will contradict (5.2). We fix a smooth non-negative function φ on M such that $\varphi = 1$ on $W'' \Subset W'$ and vanishes outside W'. By Stokes' theorem for all $i \in \mathbb{N} \cup \{+\infty\}$ and divergence-free from Lemma 5.1, we have

$$\begin{aligned}\int_M \operatorname{div}(X_i) \cdot \varphi \, d\mathrm{vol}_g &= -\int_M \langle X_i, \nabla\varphi \rangle \, d\mathrm{vol}_g \\ &= -\int_M T(\nabla f_i, \nabla\varphi) \, d\mathrm{vol}_g \\ &= \int_M f_i \cdot \langle T, \nabla^2\varphi \rangle \, d\mathrm{vol}_g.\end{aligned}$$

Thanks to (5.1) and Lemma 5.2,

$$\begin{aligned}0 \quad &< \int_M \operatorname{div}(X_\infty) \cdot \varphi \, d\mathrm{vol}_g \\ &= \int_M f_\infty \cdot \langle T, \nabla^2\varphi \rangle \, d\mathrm{vol}_g \\ &= \lim_{i\to+\infty} \int_M f_i \cdot \langle T, \nabla^2\varphi \rangle \, d\mathrm{vol}_g \\ &= \lim_{i\to+\infty} \int_M \operatorname{div}(X_i) \cdot \varphi \, d\mathrm{vol}_g.\end{aligned}$$

Since $W' \subseteq B_g(x_0, r)$, this contradicts (5.2) and thus finishes the proof. □

Proof of Theorem 1.4. For all $r > 1$, we let φ be a smooth non-negative function on M such that $\varphi = 1$ on $B_g(x_0, r)$, vanishes outside $B_g(x_0, 2r)$ and satisfies $|\nabla\varphi| \le 2r^{-1}$ on M.

By Proposition 5.4, we have

$$c_0 := \liminf_{i\to+\infty} \int_{B_g(x_0,1)} \operatorname{div}(X_i) \, d\mathrm{vol}_g > 0. \tag{5.4}$$

By (5.1), Lemma 5.2, Stokes Theorem, $0 \le T \le \mathrm{scal}$, and the choice of φ, for any $r \ge 1$ and $i \in \mathbb{N}$ we have

$$\begin{aligned}&\int_{B_g(x_0,1)} \left[\varepsilon_i \left(\frac{n}{2} - 1\right) \cdot \phi(b_i) \cdot \mathrm{scal} + \operatorname{div}(X_i)\right] d\mathrm{vol}_g \\ &\le \int_M \left[\varepsilon_i \left(\frac{n}{2} - 1\right) \cdot \phi(b_i) \cdot \mathrm{scal} + \operatorname{div}(X_i)\right] \varphi \, d\mathrm{vol}_g \\ &= \varepsilon_i \left(\frac{n}{2} - 1\right) \cdot \int_M \phi(b_i) \cdot \mathrm{scal} \cdot \varphi \, d\mathrm{vol}_g - \int_M \langle X_i, \nabla\varphi \rangle \, d\mathrm{vol}_g \\ &\le \varepsilon_i \left(\frac{n}{2} - 1\right) \cdot \int_M \phi(b_i) \cdot \mathrm{scal} \cdot \varphi \, d\mathrm{vol}_g + \frac{4}{r} \int_{A_g(x_0,r,2r)} \mathrm{scal} \, d\mathrm{vol}_g.\end{aligned}$$

The conclusion follows by letting $i \to +\infty$ for each fixed $r \ge 1$, by our choice of $c_0 > 0$. □

Finally we note that in case of $\sec > 0$ nearby x_0, the constant c_1 in Theorem 1.4 can be estimated from below by curvature lower bound over a ball centered at x_0.

Proposition 5.5. *In the setting of Theorem 1.4, if in addition* $\sec > 0$ *on* $B_g(x_0, r)$ *for* $r \in (0, 1]$*, then we might choose* c_1 *so that*

$$c_1 \ge \frac{(n-1)(n-2)}{8(1+r^2)^{3/2}} \cdot \mathrm{Vol}_g(x_0, r) \cdot \inf_{B_g(x_0,r)} \sec(g) > 0.$$

Proof. We denote $\varepsilon := \inf_{B_g(x_0,r)} \sec > 0$. In the proof of Theorem 1.4, we might choose the Busemann function b such that $b(x_0) = 0 = \inf_M b$. By the proof of [9, Lemma 3.2],

$$T_{ij} \geq \frac{1}{2}\varepsilon(n-1)(n-2)g_{ij} > 0$$

on $B_g(x_0, r)$ so that (5.1) implies

$$\operatorname{div}(X_i) \geq -o(1) + \frac{1}{2}\varepsilon(n-1)(n-2)\phi''(b_i)|\nabla b_i|^2 \geq -o(1)$$

on $B_g(x_0, r)$, as $i \to +\infty$. Furthermore by Lemma 5.2, b_i are 1-Lipschitz and hence $b_i(x) \leq b_i(x_0) + r \leq \varepsilon_i + r$ on $B_g(x_0, r)$. In particular,

$$\phi''(b_i) = (1 + b_i^2)^{-3/2} \geq \left(1 + (\varepsilon_i + r)^2\right)^{-3/2}.$$

From the choice of c_0 in (5.4), this yields

$$\begin{aligned} c_0 &\geq \liminf_{i\to+\infty} \int_{B_g(x_0,r)} \operatorname{div}(X_i)\, d\mathrm{vol}_g \\ &\geq \frac{\varepsilon(n-1)(n-2)}{2(1+r^2)^{3/2}} \cdot \liminf_{i\to+\infty} \int_{B_g(x_0,r)} |\nabla b_i|^2\, d\mathrm{vol}_g \\ &= \frac{\varepsilon(n-1)(n-2)}{2(1+r^2)^{3/2}} \cdot \mathrm{Vol}_g(x_0, r), \end{aligned}$$

since ∇b_i converges weakly to ∇b and $|\nabla b| = 1$ almost everywhere. □

6. Volume and Gap Theorem of three-manifolds under Ricci

In this section, we discuss the proof of Theorem 1.5, i.e. how vanishing of asymptotic curvature forces M to be flat under $\mathrm{Ric} \geq 0$. We will prove a slightly more general theorem, which says that if $k(x_0, r) = O(1)$ as $r \to +\infty$, then it is either $\mathrm{AVR} > 0$ or has quadratic volume growth with finite total curvature. This is analogous to Corollary 1.3.

Theorem 6.1. *Suppose (M^3, g) is a complete non-compact non-flat manifold such that $\mathrm{Ric} \geq 0$, $\mathrm{AVR}(g) = 0$ and there exists $x_0 \in M$ so that $\mathrm{scal}(x_0) > 0$ and $\beta := \limsup_{r\to+\infty} k(x_0, r) < +\infty$ where $k(x_0, r)$ is defined in (1.2), then there is $\Lambda > 1$ such that*

(i) $\Lambda^{-1} r^2 \leq \mathrm{Vol}(x_0, r) \leq \Lambda r^2$ *for all large $r > 0$ and;*

(ii) $\displaystyle\int_M \mathrm{scal}(g)\, d\mathrm{vol}_g \leq \Lambda\beta.$

We note that the limit β is indeed independent of the base point x_0. Assuming Theorem 6.1, the gap Theorem 1.5 is immediate.

Proof of Theorem 1.5 assuming Theorem 6.1. Suppose $\mathrm{AVR} > 0$, then Proposition 4.1 implies $\mathrm{AVR} = 1$ and hence (M^3, g) is flat by rigidity in volume comparison. Hence, we might assume $\mathrm{AVR} = 0$. In this case, Theorem 6.1 implies that total scalar curvature vanishes, and so it is flat. □

In the remainder of this section, we are devoted to proving Theorem 6.1. For a later purpose, we also choose $\sigma \in (0, 1)$ sufficiently small so that

$$\int_{\partial B_g(x_0,\sigma)} H^2\, dA_g < 16\pi. \tag{6.1}$$

using [36, Proposition 3.1].

As explained briefly in the introduction, we will start with observing that (M^3, g) in Theorem 6.1 must have at least quadratic volume growth.

Lemma 6.2. *Suppose (M^n, g) is a complete non-flat non-compact manifold such that $\limsup_{r\to+\infty} k(x_0, r) < +\infty$, then there is $c_1 > 0$ such that*

$$\liminf_{r\to+\infty} \frac{\mathrm{Vol}_g(x_0, r)}{r^2} \geq c_1.$$

Proof. Since $k(x_0, r)$ is bounded from above as $r \to +\infty$, we choose $R_0, C_0 > 1$ such that for all $r > R_0$, $k(x_0, r) \leq C_0$ and hence

$$\begin{aligned}\mathrm{Vol}_g(x_0, r) &\geq C_0^{-1} r^2 \cdot \int_{B_g(x_0,r)} \mathrm{scal}\, d\mathrm{vol}_g \\ &\geq C_0^{-1} r^2 \cdot \int_{B_g(x_0,1)} \mathrm{scal}\, d\mathrm{vol}_g,\end{aligned}$$

for all $r > R_0$. Since $\mathrm{scal}(x_0) > 0$, the result follows. □

We note that Lemma 6.2 is dimension independent but for convenience, we restrict to $n = 3$. Lemma 6.2 implies that (M^3, g) is p-nonparabolic for a well-chosen $p \in (1, 2]$.

Lemma 6.3. *(M^3, g) is p-nonparabolic, for $p = 4/3 \in (1, 2)$.*

Proof. By Lemma 6.2, for any $x \in M$ we have

$$\begin{aligned}\int_1^\infty \left(\frac{s}{\mathrm{Vol}_g(x, s)}\right)^{\frac{1}{\frac{4}{3}-1}} ds &= \int_1^\infty \left(\frac{s}{\mathrm{Vol}_g(x, s)}\right)^3 ds \\ &\lesssim \int_1^\infty \frac{1}{s^3}\, ds < +\infty.\end{aligned}$$

Since $\mathrm{Ric} \geq 0$, the assertion follows by [28, Proposition 5.10], see [20, Remark 4.2]. □

We consider the Dirichlet problem of the p-harmonic function on the exterior domain $M \setminus B_g(x_0, \sigma)$:

$$\begin{cases} \Delta_{\frac{4}{3}} u = 0 & \text{in } \Omega := M \setminus \overline{B_g(x_0, \sigma)} \\ u = 1 & \text{on } \partial B_g(x_0, \sigma) \\ u(x) \to 0 & \text{as } x \to \infty. \end{cases} \tag{6.2}$$

Lemma 6.4. *Under the assumption of Theorem 6.1,* (6.2) *admits a solution u such that*

(i) *$0 < u < 1$ on $M \setminus \overline{B_g(x_0, \sigma)}$;*
(ii) *$\int_\Omega |\nabla u|^{4/3}\, d\mathrm{vol}_g < +\infty$;*
(iii) *There is $C_0 > 0$ such that for $x \to \infty$,*

$$u(x) \leq C_0 \left(\int_{d_g(x,x_0)}^\infty \mathrm{Vol}(x_0, s)^{-3/4}\, ds\right)^4;$$

(iv) *u is $C^{1,\beta}$ up to $\partial\Omega$ and is smooth with non-vanishing gradient in a neighborhood of $\partial\Omega$.*

Proof. The existence of u with (ii) follows from [20, Theorem 4.1] and Lemma 6.3. The property (i) follows from the strong maximum principle of p-harmonic function. The estimate (iii) for p-Green function follows from [29, Proposition 4.4] and [23, Proposition 5.7], as pointed out in [29, p. 15], it carries over to the Dirichlet solution u. The smoothness across the boundary follows from [20, Theorem 4.1 & Appendix A]. □

It is more convenient to consider the normalized function. We denote $\rho := d_g(x_0, \cdot)$, $w := -\frac{1}{3}\log u$ and

$$\Omega_t := \overline{B_g(x_0, \sigma)} \cup \{x \in M \setminus \overline{B_g(x_0, \sigma)} : w(x) < t\},\ t > 0.$$

Lemma 6.5 (Boundary regularity and the logarithmic gradient)**.** *We have $w > 0$ on Ω, vanishes on $\partial\Omega$ and satisfies*

$$\operatorname{div}(|\nabla w|^{-2/3}\nabla w) = |\nabla w|^{4/3}.$$

For each $T > 0$, the set $\{x \in \bar{\Omega} : 0 \le w(x) \le T\}$ is compact. There is $t_0 > 0$ such that w is smooth and $|\nabla w| > 0$ throughout this set for $T = t_0$. Moreover,

$$\sup_{M\setminus\Omega} |\nabla w| < \infty, \qquad |\nabla w|(x) \le \frac{C}{\rho(x)} \quad (\rho(x) \ge 2\sigma).$$

Proof. By Lemma 6.4, there is $\delta > 0$ such that w is smooth and $|\nabla w| > 0$ on $\{\sigma \le \rho \le \sigma+\delta\}$. Since $u < 1$ in the exterior and $u \to 0$ at infinity, $\sup_{\{\rho\ge\sigma+\delta\}} u < 1$. Choose $t_0 > 0$ with $e^{-3t_0} > \sup_{\{\rho\ge\sigma+\delta\}} u$. Then

$$\{x \in M \setminus B_g(x_0, \sigma) : 0 \le w(x) \le t_0\} \subset \{x : \sigma \le \rho(x) < \sigma + \delta\}.$$

The same decay makes every $\{0 \le w \le T\}$ in the closed exterior closed and bounded, hence compact. Also $w = 0$ on $\partial B_g(x_0, \sigma)$ and $w > 0$ outside.

Direct computation gives $\operatorname{div}(|\nabla w|^{-2/3}\nabla w) = |\nabla w|^{4/3}$. For $\rho(x) \ge 2\sigma$, the ball $B_{\rho(x)/4}(x)$ lies in the exterior. [50, Theorem 1.1] therefore gives

$$|\nabla w|(x) = \tfrac{1}{3}|\nabla \log u(x)| \le \frac{4C_W}{3\rho(x)} \qquad (\rho(x) \ge 2\sigma),$$

where C_W depends only on dimension and $p = 4/3$. On the remaining compact annulus, interior and boundary C^1 regularity and positivity imply

$$\sup_{\{\sigma\le\rho\le2\sigma\}} |\nabla w| \le \frac{\max\limits_{\{\sigma\le\rho\le2\sigma\}} |\nabla u|}{3 \min\limits_{\{\sigma\le\rho\le2\sigma\}} u} < \infty.$$

Together these estimates give the global bound for $|\nabla w|$. □

Denote the regular part of each level set of w by Σ_t:

$$\Sigma_t := \{x \in M \setminus B_g(x_0, \sigma) : w(x) = t, \nabla w \ne 0.\}.$$

so that

$$\int_{\Sigma_t} \operatorname{scal}(g_{\Sigma_t}) \in 8\pi\mathbb{Z}, \tag{6.3}$$

for almost all positive t, by [3, Theorem 1.3].

We consider the following quantities similar to [2]

$$G(t) := \frac{9}{25}\int_{\Sigma_t} |\nabla w|^2, \quad F(t) := \frac{3}{5}\int_{\Sigma_t} H|\nabla w| - G(t),$$

in which

$$F(t) = \int_{\Sigma_t} \frac{H^2}{4} - \left(\frac{H}{2} - \frac{3}{5}|\nabla w|\right)^2 \le \int_{\Sigma_t} \frac{H^2}{4}.$$

From (6.1), we also have $F(0) < 4\pi$. More importantly, $F(t)$ and $G(t)$ satisfy a monotonicity formula under $\mathrm{Ric} \ge 0$.

Lemma 6.6. [2, Lemmas 2.2 and 2.3] *$F \in W^{1,1}_{loc}(0,\infty)$ and $G \in W^{2,1}_{loc}(0,\infty)$ are continuous at $t = 0$ and satisfy the following equations for a.e. $t > 0$*

$$\begin{aligned} G'(t) &= 3(G(t) - F(t)) \\ F'(t) &= -\frac{3}{5}\int_{\Sigma_t} \left[\mathrm{Ric}(\nu,\nu) + |A^\circ|^2 + |\nabla^{\Sigma_t} \ln|\nabla w||^2 + \frac{5}{2}\left(H - \frac{6}{5}|\nabla w|\right)^2\right]. \end{aligned}$$

Both F and G are nonincreasing in t,

$$0 \le G(t) \le F(t) \le F(0) < 4\pi \qquad (t \ge 0)$$

$$\int_0^\infty \left(\int_{\Sigma_t} \left(H - \frac{6}{5}|\nabla w| \right)^2 \right) dt \le \frac{2}{3} F(0).$$

Proof. The proof can be found in [2, Lemmas 2.2 and 2.3]. The integral estimate follows from integrating F' from $t = 0$ to $t = +\infty$ using $\mathrm{Ric} \ge 0$ and $F(0) < 4\pi$. □

Lemma 6.7. [1] *For almost every $t > 0$,*

$$e^{-t} \int_{\Sigma_t} |\nabla w|^{1/3} = \int_{\partial B_g(x_0,\sigma)} |\nabla w|^{1/3} = 3^{-1/3} \int_{\partial B_g(x_0,\sigma)} |\nabla u|^{1/3} > 0. \tag{6.6}$$

Moreover, F, G extend absolutely continuously to every $[0, T]$, with their values at $t = 0$ equal to the integrals over $\partial B_g(x_0, \sigma)$.

Proof. For the potential u constructed,

$$|\nabla u|^{-2/3} \nabla u = -3^{1/3} e^{-w} |\nabla w|^{-2/3} \nabla w,$$

with value zero whenever $\nabla w = 0$. [1, Proposition 2.8] gives, for almost every $\tau \in (0, 1)$,

$$\int_{\{u=\tau\}} |\nabla u|^{1/3} = \int_{\partial B_g(x_0,\sigma)} |\nabla u|^{1/3}.$$

On Σ_t, $u = e^{-3t}$ and $|\nabla u|^{1/3} = 3^{1/3} e^{-t} |\nabla w|^{1/3}$. This proves the identity. The strict positivity then follows from (iv) of Lemma 6.4. Note that Lemma 6.5 implies that the level sets Σ_t of w are regular for $t \in [0, t_0]$. Hence F and G are smooth for $t \in [0, t_0]$, together with $F \in W^{1,1}_{loc}(0, \infty)$ and $G \in W^{2,1}_{loc}(0, \infty)$ from Lemma 6.6, F, G extend absolutely continuously to every $[0, T]$. □

The next proposition provides a lower bound for the volumes of sublevel sets of the potential function.

Proposition 6.8. *There are constants $c, C > 0$ such that for all $T > 1$,*

$$\mathrm{Vol}_g(\Omega_T) \ge c \exp\left[3T - C \int_{\Omega_T \setminus B_g(x_0,\sigma)} \mathrm{scal}(g) |\nabla w| \, d\mathrm{vol}_g \right] \qquad (T > 1). \tag{6.7}$$

After increasing the fixed radius r_0 if necessary, there are constants $c, C > 0$, independent of r, T, such that if $T > 1$, $r \ge r_0$, $\Omega_T \subset B_g(x_0, r)$

$$\mathrm{Vol}_g(x_0, r) \ge c \exp\left[3T - C \int_{B_g(x_0,r) \setminus B_g(x_0,r_0)} \frac{\mathrm{scal}(g)}{\rho} \, d\mathrm{vol}_g \right]. \tag{6.8}$$

Proof. By Lemma 6.6, $F - G/2 \ge G/2 \ge 0$. Thanks to Lemma 6.7 and monotonicity from Lemma 6.6, $G(t) > 0$ for all t and thus $F - G/2 > 0$.

By Lemma 6.7, $\log(F - G/2)$ is absolutely continuous. We first obtain a differential inequality for this logarithm. The Gauss equation and the definition of F give

$$\begin{aligned} \int_{\Sigma_t} \left[\mathrm{scal}(g) + \frac{1}{2} \left(H - \frac{6}{5}|\nabla w| \right)^2 \right] &= \int_{\Sigma_t} \mathrm{scal}(g_{\Sigma_t}) - 2F(t) \\ &\quad + \int_{\Sigma_t} \left[2\mathrm{Ric}(\nu, \nu) + |A^\circ|^2 \right]. \end{aligned} \tag{6.9}$$

By Cauchy–Schwarz, we have

$$(F-G)^2 = \left[\frac{3}{5}\int_{\Sigma_t} |\nabla w|\left(H-\frac{6}{5}|\nabla w|\right)\right]^2 \le G\int_{\Sigma_t}\left(H-\frac{6}{5}|\nabla w|\right)^2. \tag{6.10}$$

Claim 6.1.

$$\frac{d}{dt}\log\left(F-\frac{1}{2}G\right)+\frac{3}{2} \le \frac{3}{8\pi-2F(0)}\int_{\Sigma_t}\left[\mathrm{scal}(g)+\frac{1}{2}\left(H-\frac{6}{5}|\nabla w|\right)^2\right].$$

Proof of Claim 6.1. By (6.3), for almost every t one of the following holds:

(I) $\displaystyle\int_{\Sigma_t}\mathrm{scal}(g_{\Sigma_t}) \le 0$ or

(II) $\displaystyle\int_{\Sigma_t}\mathrm{scal}(g_{\Sigma_t}) \ge 8\pi$.

In case of (I), (6.9) and $\mathrm{scal}(g) \ge \mathrm{Ric}(\nu,\nu)$ imply

$$\int_{\Sigma_t}\left[\mathrm{Ric}(\nu,\nu)+|A^\circ|^2\right] \ge 2F+\frac{1}{2}\int_{\Sigma_t}\left(H-\frac{6}{5}|\nabla w|\right)^2.$$

Substituting into the formula of F' in Lemma 6.6 gives

$$F' \le -\frac{6}{5}F-\frac{9}{5}\int_{\Sigma_t}(H-\frac{6}{5}|\nabla w|)^2.$$

Thus $G'=3(G-F)$ and (6.10) yield

$$\begin{aligned}&\left(F-\frac{1}{2}G\right)'+\frac{3}{2}\left(F-\frac{1}{2}G\right)\\ &= F'+3F-\frac{9}{4}G\\ &\le \frac{9}{5}F-\frac{9}{4}G-\frac{9}{5}\frac{(F-G)^2}{G} = -\frac{9}{5}\frac{(F-\frac{3}{2}G)^2}{G} \le 0.\end{aligned}$$

This proves the inequality in case (I).

In case of (II), (6.9) and $F \le F(0) < 4\pi$ give

$$\int_{\Sigma_t}\left[\mathrm{scal}(g)+\frac{1}{2}\left(H-\frac{6}{5}|\nabla w|\right)^2\right] \ge 8\pi-2F(0) > 0. \tag{6.11}$$

In this case it suffices to use $F' \le 0$ and $G \ge 0$ to show that

$$\frac{d}{dt}\log\left(F-\frac{1}{2}G\right)+\frac{3}{2} = \frac{F'+3F-\frac{9}{4}G}{F-\frac{1}{2}G} \le 3.$$

Together with (6.11), this completes the proof. □

We integrate $\log(F-\frac{1}{2}G)$ to conclude

$$\begin{aligned}&\left[\log\left(F(s)-\frac{1}{2}G(s)\right)+\frac{3}{2}s\right]\Big|_{s=0}^{s=t}\\ &\le \frac{3}{8\pi-2F(0)}\int_0^t\int_{\Sigma_s}\left[\mathrm{scal}(g)+\frac{1}{2}\left(H-\frac{6}{5}|\nabla w|\right)^2\right]ds\\ &=: \mathbf{I}+\mathbf{II}.\end{aligned}$$

For **I**, we use coarea to give

$$\int_0^t \left(\int_{\Sigma_s} \mathrm{scal}(g) \right) ds = \int_{\Omega_t \setminus B_g(x_0,\sigma)} \mathrm{scal}(g) \cdot |\nabla w| \, d\mathrm{vol}_g$$

while

$$\int_0^t \left[\int_{\Sigma_s} \left(H - \frac{6}{5} |\nabla w| \right)^2 \right] ds \leq \frac{2}{3} F(0),$$

using Lemma 6.6. Exponentiating and using $G \leq 2(F - G/2)$ now gives

$$G(t) \leq C_0 \exp \left[-\frac{3}{2} t + C_0 \int_{\Omega_t \setminus B_g(x_0,\sigma)} \mathrm{scal}(g) |\nabla w| \, d\mathrm{vol}_g \right]. \tag{6.12}$$

To pass to volume, Hölder's inequality on the regular part of Σ_t gives

$$\int_{\Sigma_t} |\nabla w|^{1/3} \leq \left(\int_{\Sigma_t} |\nabla w|^2 \right)^{4/9} \left(\int_{\Sigma_t} |\nabla w|^{-1} \right)^{5/9}.$$

The flux formula (6.6) makes the left side $e^t \int_{\partial B_g(x_0,\sigma)} |\nabla w|^{1/3}$, and $\int_{\Sigma_t} |\nabla w|^2 = (25/9) G(t)$. Rearranging and using (6.12), we obtain

$$\begin{aligned} \int_{\Sigma_t} |\nabla w|^{-1} &\geq c e^{9t/5} G(t)^{-4/5} \\ &\geq c \exp \left[3t - C \int_{\Omega_t \setminus B_g(x_0,\sigma)} \mathrm{scal}(g) |\nabla w| \, d\mathrm{vol}_g \right]. \end{aligned}$$

The fixed boundary flux is positive; the exponent is $9/5 + (4/5)(3/2) = 3$. If the left side is infinite, the inequality is automatic. Coarea on $\{|\nabla w| > 0\}$ and $\Omega_t \subset \Omega_T$ for $t < T$ give

$$\begin{aligned} \mathrm{Vol}_g(\Omega_T) &\geq \int_{T-1}^T \left[\int_{\Sigma_t} |\nabla w|^{-1} \right] dt \\ &\geq c \exp \left[-C \int_{\Omega_T \setminus B_g(x_0,\sigma)} \mathrm{scal}(g) |\nabla w| \, d\mathrm{vol}_g \right] \int_{T-1}^T e^{3t} \, dt. \end{aligned}$$

Since $\int_{T-1}^T e^{3t} \, dt = (1 - e^{-3}) e^{3T}/3$, this proves (6.7). This argument discards any volume of the critical set.

Finally, choose $r_0 \geq 2\sigma$ so that $|\nabla w| \leq C/\rho$ for $\rho \geq r_0$, using Lemma 6.5. If $\Omega_T \subset B_g(x_0, r)$,

$$\begin{aligned} \int_{\Omega_T \setminus B_g(x_0,\sigma)} \mathrm{scal}(g) |\nabla w| \, d\mathrm{vol}_g \quad \leq \quad & \left(\sup_{\{\sigma \leq \rho \leq r_0\}} |\nabla w| \right) \int_{B_g(x_0,r_0)} \mathrm{scal}(g) \, d\mathrm{vol}_g \\ & + C \int_{B_g(x_0,r) \setminus B_g(x_0,r_0)} \frac{\mathrm{scal}(g)}{\rho} \, d\mathrm{vol}_g. \end{aligned}$$

The first term is a fixed finite constant by Lemma 6.5 and smoothness of $\mathrm{scal}(g)$ on the compact closed ball. Absorbing it into c in (6.7) proves (6.8). □

For r_0 from Proposition 6.8, we define

$$m(r) := \inf_{s \geq r} \frac{\mathrm{Vol}_g(x_0, s)}{s^2}, \quad \forall r \geq r_0.$$

By Lemma 6.2, we might assume r_0 large so that $k(x_0, r) \leq C_1 := \beta + 1$ for $r > r_0$ and

$$0 < c \leq m(r) \leq \frac{\mathrm{Vol}_g(x_0, r)}{r^2}, \qquad m(r_1) \leq m(r_2) \tag{6.13}$$

for $r_2 \geq r_1 \geq r_0$. Moreover, by $\mathrm{scal} \geq 0$, for all $s > r > r_0$

$$\begin{aligned} \int_{B_g(x_0,r)} \mathrm{scal}\, d\mathrm{vol}_g &\leq \int_{B_g(x_0,s)} \mathrm{scal}\, d\mathrm{vol}_g \\ &= k(x_0,s) \cdot \frac{\mathrm{Vol}_g(x_0,s)}{s^2} \end{aligned} \tag{6.14}$$

Taking the infimum over $s > r$ shows that for all $r \geq r_0$,

$$\int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g \leq C_1 m(r). \tag{6.15}$$

Since $\mathrm{AVR}(g) = 0$, by (6.14) and Bishop-Gromov comparison, for any $\varepsilon > 0$, there is $r_\varepsilon > r_0$ such that for all $r > r_\varepsilon$,

$$\frac{1}{r} \int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g \leq \varepsilon. \tag{6.16}$$

We next show that the error term appearing in Proposition 6.8 is negligible if $m(r) \to +\infty$.

Lemma 6.9. *Under the assumptions of Theorem 6.1, if in addition $m(r) \to \infty$, then*

$$\frac{1}{\log m(r)} \int_{B_g(x_0,r)\setminus B_g(x_0,r_0)} \rho^{-1} \cdot \mathrm{scal}(g)\, d\mathrm{vol}_g \longrightarrow 0.$$

Proof. By assumption, $\log m(r) > 0$ for all large r. The hypothesis $m(r) \to \infty$ ensures that $\log m(r) > 0$ for all sufficiently large r. For $r_0 < \rho < r$, the identity $1/\rho = 1/r + \int_\rho^r s^{-2}\, ds$ and Tonelli's theorem give

$$\begin{aligned} &\int_{B_g(x_0,r)\setminus B_g(x_0,r_0)} \frac{\mathrm{scal}(g)}{\rho}\, d\mathrm{vol}_g \\ &\leq \frac{1}{r} \int_{B_g(x_0,r)} \mathrm{scal}(g)\, d\mathrm{vol}_g + \int_{r_0}^r \frac{1}{s^2} \left(\int_{B_g(x_0,s)} \mathrm{scal}(g)\, d\mathrm{vol}_g \right) ds. \end{aligned}$$

The first term is bounded by (6.16). For the second, we use (6.15) instead, giving

$$\begin{aligned} &\int_{r_0}^r \frac{1}{s^2} \left(\int_{B_g(x_0,s)} \mathrm{scal}(g)\, d\mathrm{vol}_g \right) ds \\ &\leq \int_{r_0}^{r_\varepsilon} \frac{1}{s^2} \left(\int_{B_g(x_0,s)} \mathrm{scal}(g)\, d\mathrm{vol}_g \right) ds + \int_{r_\varepsilon}^r s^{-2} \cdot \min\{C_1 m(s), \varepsilon s\}\, ds \end{aligned}$$

We compute the second term explicitly, obtaining

$$\begin{aligned} \int_{r_\varepsilon}^r s^{-2} \cdot \min\{C_1 m(s), \varepsilon s\}\, ds &\leq \int_{r_\varepsilon}^{A_0 m(r)/\varepsilon} \frac{\varepsilon}{s}\, ds + \int_{C_1 m(r)/\varepsilon}^\infty \frac{C_1 m(s)}{s^2}\, ds \\ &= \varepsilon \log \frac{C_1 m(r)}{\varepsilon r_\varepsilon} + \varepsilon. \end{aligned}$$

The result follows by first letting $r \to +\infty$ and followed by $\varepsilon \to 0$. □

We now prove that the total curvature is bounded.

Proposition 6.10. *Under the assumption of Theorem 6.1, we have*

$$\sup_{r>0} m(r) < \infty, \qquad 0 < \int_M \mathrm{scal}(g)\, d\mathrm{vol}_g < \infty.$$

Proof. By (6.15), it suffices to bound m. If m were unbounded, then (6.13) would give

$$m(r) \longrightarrow \infty, \qquad \frac{\mathrm{Vol}_g(x_0, r)}{r^2} \geq m(r) \longrightarrow \infty. \tag{6.17}$$

We now derive a contradiction using Proposition 6.8. By Lemma 6.4 for $\rho(x) \to +\infty$,

$$\begin{aligned} u(x) &\leq C \left(\int_{\rho(x)}^{\infty} \frac{ds}{\mathrm{Vol}_g(x_0, s)^{3/4}} \right)^4 \\ &\leq \frac{C}{m(\rho(x))^3} \left(\int_{\rho(x)}^{\infty} s^{-3/2}\, ds \right)^4 \\ &= \frac{16C}{m(\rho(x))^3 \rho(x)^2}. \end{aligned}$$

The second line uses $\mathrm{Vol}_g(x_0, s) \geq m(\rho(x))s^2$ for every $s \geq \rho(x)$. This implies

$$w(x) \geq \frac{2}{3} \log \rho(x) + \log m(\rho(x)) - C_0. \tag{6.18}$$

The continuous function $r \mapsto \mathrm{Vol}_g(x_0, r)/r^2$ tends to infinity, so it attains a minimum on every closed interval $[j, \infty)$ with integer $j \geq r_0$. To see this directly, its values eventually exceed $\mathrm{Vol}_g(x_0, j)/j^2 + 1$; a minimum on the remaining compact interval is therefore also a minimum on the whole tail. Choose a minimizing point $r_j \geq j$. For every $s \geq r_j$ we then have $\mathrm{Vol}_g(x_0, s)/s^2 \geq \mathrm{Vol}_g(x_0, r_j)/r_j^2$, and hence

$$\frac{\mathrm{Vol}_g(x_0, r_j)}{r_j^2} = m(r_j), \qquad r_j \longrightarrow \infty, \qquad m(r_j) \longrightarrow \infty. \tag{6.19}$$

Define the associated levels by

$$T_j = \frac{2}{3} \log r_j + \log m(r_j) - C_0 - 1.$$

Equations (6.19) give $T_j \to \infty$. For all large j, (6.13) and (6.18) imply

$$\rho(x) \geq r_j \quad \Longrightarrow \quad w(x) \geq \frac{2}{3} \log r_j + \log m(r_j) - C_0 = T_j + 1.$$

Thus $\Omega_{T_j} \subseteq B_g(x_0, r_j)$. Since

$$e^{3T_j} = e^{-3C_0 - 3} r_j^2 m(r_j)^3,$$

substitution into (6.8) gives

$$r_j^2 m(r_j) = \mathrm{Vol}_g(x_0, r_j) \geq c\, r_j^2 m(r_j)^3 \exp\left[-C \int_{B_g(x_0, r_j) \setminus B_g(x_0, r_0)} \frac{\mathrm{scal}(g)}{\rho}\, d\mathrm{vol}_g \right].$$

Rearrangement gives

$$2 \log m(r_j) \leq C + C \int_{B_g(x_0, r_j) \setminus B_g(x_0, r_0)} \frac{\mathrm{scal}(g)}{\rho}\, d\mathrm{vol}_g.$$

and hence,

$$2 \leq \frac{C}{\log m(r_j)} + \frac{C}{\log m(r_j)} \int_{B_g(x_0, r_j) \setminus B_g(x_0, r_0)} \frac{\mathrm{scal}(g)}{\rho}\, d\mathrm{vol}_g \longrightarrow 0,$$

by Lemma 6.9. This contradicts (6.17) and thus m is bounded. □

Finally, we show that the volume can grow at most quadratically.

Proposition 6.11. *Under the assumption of Theorem 6.1, we have*

$$\limsup_{r\to+\infty}\frac{\mathrm{Vol}_g(x_0,r)}{r^2}<+\infty.$$

Proof. By Lemma 6.5, we already have

$$\sup_{M\setminus\Omega}|\nabla w|<\infty.$$

Together with the assumed finite scalar integral, this implies

$$\int_{M\setminus\Omega}\mathrm{scal}(g)|\nabla w|\,d\mathrm{vol}_g\le\left(\sup_{M\setminus\Omega}|\nabla w|\right)\int_M\mathrm{scal}(g)\,d\mathrm{vol}_g<\infty$$

by Proposition 6.10.

Substituting this uniform bound into (6.7) gives

$$\mathrm{Vol}_g(\Omega_T)\ge ce^{3T}, \tag{6.20}$$

for all $T\to+\infty$, where $c>0$.

From Lemma 6.4, for all large $\rho(x)$

$$u(x)\le C_0\left(\int_{\rho(x)}^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^4. \tag{6.21}$$

For a large radius r, set

$$T(r):=-\frac{1}{3}\log\left[C_0\left(\int_r^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^4\right]-1$$

so that Lemma 6.2 implies $T\to\infty$ as $r\to\infty$.

Hence if $\rho(x)\ge r$, then $w(x)\ge T(r)+1$ by (6.21). This proves $\Omega_T\subset B_g(x_0,r)$ for large r. Applying (6.20) gives

$$\mathrm{Vol}_g(x_0,r)\ge ce^{3T}=\frac{ce^{-3}}{C_0\left(\int_r^{\infty}\mathrm{Vol}_g(x_0,s)^{-3/4}\,ds\right)^4}.$$

Thus there is a fixed $c_2>0$ such that

$$\mathrm{Vol}_g(x_0,r)\cdot\left(\int_r^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^4\ge c_2, \tag{6.22}$$

for all r large.

The fundamental theorem of calculus therefore gives

$$\begin{aligned}\frac{d}{dr}\left(\int_r^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^{-2}&=2\left[\mathrm{Vol}_g(x_0,r)\left(\int_r^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^4\right]^{-3/4}\\&\le 2c_2^{-3/4}.\end{aligned}$$

Choose a fixed $r_2\ge r_0$ so that (6.22) holds for $r\ge r_2$. Integration yields

$$\begin{aligned}\left(\int_r^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^{-2}&\le\left(\int_{r_2}^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^{-2}+2c_2^{-3/4}(r-r_2)\\&\le\left[\frac{1}{r_2}\left(\int_{r_2}^{\infty}\frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}}\right)^{-2}+2c_2^{-3/4}\right]r\end{aligned}$$

for $r\ge r_2$.

That said, for r sufficiently large, we have

$$\int_r^\infty \frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}} \geq C_1 r^{-1/2}.$$

On the other hand, using Lemma 6.2, one has

$$\int_r^\infty \frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}} \leq C_2 r^{-\frac{1}{2}}.$$

For a constant $A > 1$, we obtain

$$\frac{(A-1)r}{\mathrm{Vol}_g(x_0,r)^{3/4}} \geq \int_r^{Ar} \frac{ds}{\mathrm{Vol}_g(x_0,s)^{3/4}} \geq C_1 r^{-1/2} - A^{-\frac{1}{2}} C_2 r^{-\frac{1}{2}}.$$

Choosing large enough A, this gives us the desired volume upper bound of $\mathrm{Vol}_g(x_0,r)$. □

Now we have all ingredients for Theorem 6.1.

Proof of Theorem 6.1. The volume lower bound follows from Lemma 6.2, the finiteness of total scalar curvature follows from Proposition 6.10, while the quadratic volume upper bound follows from Proposition 6.11. It remains to show that the total scalar curvature is bounded by a multiple of $\beta := \limsup_{r\to+\infty} k(x_0,r)$. It follows from refining (6.14). For any $\varepsilon > 0$, there is $r_\varepsilon > 0$ such that for all $r_\varepsilon > r$,

$$\begin{aligned}\int_{B_g(x_0,r)} \mathrm{scal}\, d\mathrm{vol}_g &\leq \int_{B_g(x_0,r_\varepsilon)} \mathrm{scal}\, d\mathrm{vol}_g \\ &\leq k(x_0,r_\varepsilon)\cdot \frac{\mathrm{Vol}_g(x_0,r_\varepsilon)}{r_\varepsilon^2} \leq (\beta+\varepsilon)\cdot\Lambda.\end{aligned}$$

Result follows by letting $\varepsilon \to 0$.

□

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