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# On Gaussian curvature flow

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**Abstract**

The current article is to study the solvability of Nirenberg problem on $\mathbb{S}^2$ through the so-called Gaussian curvature flow. We aim to propose a unified method to treat the problem for candidate functions without sign restriction and non-degenerate assumption. As a first step, we reproduce the following statement: suppose the critical points of a smooth function $f$ with positive critical values are non-degenerate. Then the required solution exists, if the difference between the number of the local maximum points with positive values and the number of the saddle points with positive critical values as well as negative Laplace is not equal to 1. This statement has been proved for nearly thirty years through different methods.


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## 1. Introduction

Current article aims to treat the classical well-known Nirenberg problem. It is commonly known that the problem is equivalent to solving the differential equation on $\mathbb{S}^2$

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$$-\Delta_{\mathbb{S}^2} u + 1 = f e^{2u} \qquad \text{on } \mathbb{S}^2, \tag{1.1}$$

with $f$ a given smooth function on $\mathbb{S}^2$.

Our favorable method is to adopt the negative gradient flow with new normalization. The global existence of the flow is always true as long as $f$ is a smooth function which is positive somewhere. Difficulty lies on the convergence as everyone expected. Through the tough effort, our finding can be stated as follows:

**Theorem 1.1.** *Let $f \in C^\infty(\mathbb{S}^2)$ be a smooth function and positive somewhere on $\mathbb{S}^2$. Assume the following conditions hold true:*

*(a) Critical points of $f$ are isolated in the region where $f > 0$;*
*(b) $|\nabla f(x)|^2_{\mathbb{S}^2} + |\Delta_{\mathbb{S}^2} f(x)|^2 \neq 0$ whenever $f(x) > 0$;*
*(c) let $p$ be the number of local maximum points of $f$ with positive values and $q$ be the number of the saddle points with positive values and negative Laplace, $p - q \neq 1$.*

*Then there exists some conformal metric $g = e^{2u} g_{\mathbb{S}^2}$ such that the Gaussian curvature of $g$ is equal to the given function $f$.*

In order to solve the equation, it is known that $f$ must be positive somewhere. Other famous obstruction to the existence of solutions to PDE is the so-called Kazdan-Warner condition

$$\int_{\mathbb{S}^2} X(K) e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} = 0,$$

where $X$ is any conformal vector field on $\mathbb{S}^2$. It is this obstruction that makes the problem interesting to study and deserving further investigation.

Let us briefly recall the development on this problem. Surely it has been extensively studied in the literature with various assumptions on $f$. For curvature candidate with certain symmetries, the answer is relatively complete. A pioneer work by J. Moser [24] states that an even function on $\mathbb{S}^2$ is the Gaussian curvature of a conformal metric of the round metric $g_{\mathbb{S}^2}$ if and only if it is positive somewhere. For rotational symmetric function $K$ on $\mathbb{S}^2$, precisely, $K = K(x^3)$ for $x = (x^1, x^2, x^3) \in \mathbb{S}^2$, X. Xu and P. Yang [28] established that a sufficient condition of the solvability of Eq. (1.1) is that $|K'| + |K''| \neq 0$, $K$ is positive somewhere and $K''$ changes signs in the region where $K > 0$. Later W. Chen and C. Li showed that Xu-Yang's sufficient condition is indeed necessary, see [13, Corollary 1]. For positive non-symmetric curvature candidates, A. Chang and P. Yang developed the degree theory in [9,10], readers are also referred to A. Chang, M. Gursky and P. Yang [11], K. Chang and J. Liu [8], W. Chen and W. Ding [12] etc and references therein.

There is no exception that the compactness for the approximate solutions has to address. Advantage of the flow approach is that we have equations to study, better than the sequence. For the study of the compactness of the solutions of the Nirenberg problem and its analogues, much effort has been made by H. Brezis and F. Merle [4], H. Brezis, Y. Li and I. Shafrir [6], Y. Li and I. Shafrir [21] etc. In some sense, new ideas in the compactness analysis often reshape the figure of the problem.

For general smooth function $f$ which may change its sign, about thirty years ago, Z. Han [18] considered the problem through the study of the generalized Moser theory for the energy functional on the open set in an infinitely dimensional Hilbert space. Later K. Chang and J. Liu

[7] applied their generalized Moser theory to reprove the statement as was pointed out that Han's argument might be incomplete.

In 1988, R. Hamilton [17] introduced Ricci flow on surfaces which is a conformal flow. After that, the scalar curvature flow has been adopted to study the Yamabe problem and so as for the prescribing scalar curvature problem. In 2003, S. Brendle [3] developed a negative gradient flow to study the prescribed $Q$-curvature problem on closed manifolds. Thereafter, the flow approach has been widely applied in conformal geometry. In [26], M. Struwe adopted such a flow to investigate the Nirenberg problem in the way that he was able to control the blow-up path and to recover an early result of A. Chang and P. Yang in [10]. Shortly after that, A. Malchiodi and M. Struwe [23] introduced $Q$-curvature flow on $\mathbb{S}^4$ and developed Morse theory part in the flow setting. Later, their result was extended by X. Chen and X. Xu [16] to all even dimensions, and [15] for the perturbation result of prescribed scalar curvature problem on $\mathbb{S}^n$. Aforementioned works are all concerned with positive Morse function as the curvature candidate in various prescribed curvature problems.

Later L. Ma-M. Hong [22] and P.K. Ho [19] respectively studied this question by following Struwe's argument. The former deals with the case $f$ is non-negative with indicating that they also can treat the case allowing $f$ to be negative somewhere. The latter treats the more general equation in similar setting and does allow $f$ to be the same as ours with an extra condition (Kazdan-Warner condition) although we doubt its necessary. Our work is in the same flavor, but it is more constructive than Han's as well as Chang and Liu's. The difference to the Gaussian curvature flow previously used is minor, namely we adopt the slightly different normalization which seems more suitable to the case that $f$ may be negative somewhere and hope it can be used to treat degenerate curvature candidates. Therefore, to set the stage and for self-contained, we have to re-develop the whole program in our setting. The trouble on controlling the normalized coefficient costs the length of the paper. We should point out that many key estimates either have been existed or easily derived through the similar ideas from the existing work.

Overall, in this part of the program, we just recover the most updated work on the existence. To compare, our method is different from Han [18] as well as Chang and Liu [7]. In our view, our method is more constructive and elementary. As it is known that Brendle's method does not apply to the case $\mathbb{S}^2$, Struwe's work [26] just covers the special situation ($f$ is positive and $p \geq 2, q = 0$). L. Ma and M. Hong [22] did similar work as Struwe's one just relaxing the condition on $f$ to be allowed to vanish or negative somewhere. Either Xu-Yang [28] or Chen-Li [13] just did for rotationally symmetric case.

This paper is organized as follows. In Section 2, we introduce a Gaussian curvature for a prescribed sign-changing function $f$. In Sections 3/Appendix A and 4, we establish short time and long time existences of the flow. We should point out that local existence has been often cited as well-known or refer to classical result for parabolic equations. However, to be self-contained, we provide the detailed argument in the Appendix A. The normalized flow is presented in Section 4. In Section 5, we show the asymptotic behaviors of the Gaussian curvature of the flow metric, i.e. the $L^p$ and $H^1$ convergences. Next we show in Section 6 that the normalized flow has compactness, hence it converges. In the rest of two sections, we try to transfer the compactness on the normalized metrics to its original ones. Here we adopt a contradiction argument: if the curvature candidate cannot be realized as the Gaussian curvature of any conformal metric, then, in Section 7, we analyze the finite-dimensional dynamics, eventually show that the flow concentrates at precisely one critical point $Q$ of $f$, with $f(Q) > 0$ and $\Delta_{\mathbb{S}^2} f(Q) \leq 0$. With this preparation, finally, in Section 8, we complete the Morse theory argument and reach a contradiction with condition ($c$) in Theorem 1.1. This contradiction shows that there must be a conformal metric of $g_{\mathbb{S}^2}$ having its Gaussian curvature as the given function $f$.

## 2. The flow

Suppose that $f$ is a smooth function on $\mathbb{S}^2$ such that $\int_{\mathbb{S}^2} f \mathrm{d}\mu_{\mathbb{S}^2} > 0$. We introduce the Gaussian curvature flow on $\mathbb{S}^2$: A flow metric $g(t) = e^{2u(t)} g_{\mathbb{S}^2}$ satisfies

$$\partial_t g = 2(\alpha(t) f - K)g \quad \text{and} \quad g(0) = e^{2u_0} g_{\mathbb{S}^2},$$

where $K = K_g$ is the Gaussian curvature of the flow metric $g(t)$ and the normalized coefficient

$$\alpha(t) = \frac{\int_{\mathbb{S}^2} f K \mathrm{d}\mu_g}{\int_{\mathbb{S}^2} f^2 \mathrm{d}\mu_g} \quad \text{for all} \ \ t \geq 0. \tag{2.1}$$

Here $\mathrm{d}\mu_g = e^{2u} \mathrm{d}\mu_{\mathbb{S}^2}$. In terms of the conformal factor, we have

$$\partial_t u = \alpha(t) f - K \quad \text{and} \quad u(0) = u_0. \tag{2.2}$$

In sequent, most of time we use the flow equation (2.2).

Notice that the Gaussian curvature of the conformal metric satisfies the relation:

$$-\Delta_{\mathbb{S}^2} u + 1 = K e^{2u} \quad \text{on} \ \ \mathbb{S}^2. \tag{2.3}$$

By the equation (2.2) and normalized factor $\alpha$, (2.1), there holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\left( \int_{\mathbb{S}^2} f \mathrm{d}\mu_g \right) = 2 \int_{\mathbb{S}^2} f u_t \mathrm{d}\mu_g = 2 \int_{\mathbb{S}^2} f(\alpha f - K) \mathrm{d}\mu_g = 0.$$

Thus, without loss of generality, we select the initial datum $u_0 \in C^\infty(\mathbb{S}^2)$ (e.g., see Remark 8.1 below for one way of this construction) such that

$$\int_{\mathbb{S}^2} f \mathrm{d}\mu_{g(t)} = \int_{\mathbb{S}^2} f \mathrm{d}\mu_{g(0)} = 4\pi \quad \text{for all} \ \ t \geq 0. \tag{2.4}$$

Define the average

$$\bar{u} := \fint_{\mathbb{S}^2} u \mathrm{d}\mu_{\mathbb{S}^2} = \frac{1}{4\pi} \int_{\mathbb{S}^2} u \mathrm{d}\mu_{\mathbb{S}^2}$$

and the energy

$$E[u] := \fint_{\mathbb{S}^2} (|\nabla u|^2_{\mathbb{S}^2} + 2u) \mathrm{d}\mu_{\mathbb{S}^2}.$$

**Lemma 2.1.** *On any finite time interval where the solution to the flow exists, there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t}E[u] = -2 \fint_{\mathbb{S}^2} (\alpha f - K)^2 \mathrm{d}\mu_g.$$

**Proof.** By Eqs. (2.2) and (2.3), a direct computation, together with the normalized condition, yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}E[u] =& 2 \fint_{\mathbb{S}^2} \langle \nabla u_t, \nabla u\rangle_{\mathbb{S}^2} + u_t \mathrm{d}\mu_{\mathbb{S}^2} \\ =& -2 \fint_{\mathbb{S}^2} u_t(\Delta_{\mathbb{S}^2} u - 1)\mathrm{d}\mu_{\mathbb{S}^2} \\ =& 2 \fint_{\mathbb{S}^2} K(\alpha f - K)e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \\ =& -2 \fint_{\mathbb{S}^2} (\alpha f - K)^2 \mathrm{d}\mu_g\end{aligned}$$

as desired. □

As a direct consequence of Lemma 2.1, $E[u]$ is non-increasing for $t \geq 0$. In particular, for any finite $T > 0$, if the flow exists on $[0, T)$, there holds

$$E[u(T)] + 2 \int_0^T \fint_{\mathbb{S}^2} (\alpha f - K)^2 \mathrm{d}\mu_g \mathrm{d}t = E[u_0].$$

The Onofri-Moser-Trudinger's inequality (see [25]) states that for any $u \in H^1(\mathbb{S}^2)$,

$$\fint_{\mathbb{S}^2} (|\nabla u|^2_{\mathbb{S}^2} + 2u)\mathrm{d}\mu_{\mathbb{S}^2} \geq \log \left( \fint_{\mathbb{S}^2} e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \right).$$

For all $t \geq 0$, there holds

$$4\pi = \int_{\mathbb{S}^2} f e^{2u(t)} \mathrm{d}\mu_{\mathbb{S}^2} \leq \max f \int_{\mathbb{S}^2} e^{2u(t)} \mathrm{d}\mu_{\mathbb{S}^2}. \tag{2.5}$$

Thus, the volume of $g(t)$ is uniformly bounded from below and from above, i.e.

$$\frac{1}{\max f} \leq \fint_{\mathbb{S}^2} e^{2u(t)} \mathrm{d}\mu_{\mathbb{S}^2} \leq e^{E[u_0]}. \tag{2.6}$$

By Eq. (2.6), we have

$$1 = \left( \fint_{\mathbb{S}^2} f \mathrm{d}\mu_g \right)^2 \leq \fint_{\mathbb{S}^2} f^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \fint_{\mathbb{S}^2} e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \leq e^{E[u_0]} \fint_{\mathbb{S}^2} f^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2}.$$

Thus

$$e^{-E[u_0]} \leq \fint_{\mathbb{S}^2} f^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \leq (\max |f|)^2 e^{E[u_0]}. \tag{2.7}$$

Hence, Onofri-Moser-Trudinger's inequality yields, as long as the solution exists on $[0, T]$, for any $t \in [0, T]$,

$$E[u(t)] \geq -\log(\max f),$$

whence,

$$\int_0^t \int_{\mathbb{S}^2} (\alpha f - K)^2 \mathrm{d}\mu_g \mathrm{d}t \leq 2\pi E[u_0] + 2\pi \log(\max f).$$

If the flow equation (2.2) globally exists in time, then letting $T \to \infty$ we obtain

$$\int_0^\infty \int_{\mathbb{S}^2} (\alpha f - K)^2 \mathrm{d}\mu_g \mathrm{d}t \leq 2\pi E[u_0] + 2\pi \log(\max f). \tag{2.8}$$

## 3. Short time existence

As was pointed out in the introduction, the short time existence can be quoted as the standard parabolic theory. In [3], Brendle proved the short time existence for higher order conformal flow. In general, Huisken and Polden ([20]) provided a rough proof of local time existence. Both proofs are based on the parabolic theory. The main idea of our proof is to use the original local existence for linear parabolic theory. The way to treat the nonlinear terms seems interesting. To be self-contained, we wish to provide the detailed proof of the local time existence for the flow in the appendix.

## 4. Long time existence

By now, it is rather a well known fact (see [10, Proposition 2.2]) that, for every smooth family of metric $g(t) = e^{2u(t)} g_{\mathbb{S}^2}$ there exists a family of conformal diffeomorphisms $\varphi = \varphi(t) : \mathbb{S}^2 \to \mathbb{S}^2$, smoothly depending on $t$, such that

$$\int_{\mathbb{S}^2} x \mathrm{d}\mu_h = 0, \qquad \text{for all } t \geq 0, \tag{4.1}$$

where $h=\varphi^* g = e^{2v} g_{\mathbb{S}^2}$ is called a normalized metric, and

$$v = u\circ\varphi + \frac{1}{2}\log(\det(\mathrm{d}\varphi)).$$

By conformal invariance of the energy, the area and (2.6), there hold $E[v]=E[u]$ and

$$\frac{1}{\max f} \le \fint_{\mathbb{S}^2} e^{2u(t)}\mathrm{d}\mu_{\mathbb{S}^2} = \fint_{\mathbb{S}^2} e^{2v(t)}\mathrm{d}\mu_{\mathbb{S}^2} \le e^{E[u_0]}.$$

The important ingredient of the argument is the following inequality, whose proof is omitted.

**Proposition 4.1** *(Aubin [2]). For any $\epsilon>0$, there exists a positive constant $C_\epsilon$ such that for every $v\in H^1(\mathbb{S}^2)$ with $\int_{\mathbb{S}^2} xe^{2v}\mathrm{d}\mu_{\mathbb{S}^2}=0$,*

$$\fint_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2} \le C_\epsilon \exp\left( (\frac{1}{2}+\epsilon)\fint_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} + 2\fint_{\mathbb{S}^2} v\mathrm{d}\mu_{\mathbb{S}^2} \right).$$

By Eq. (2.5), lower bound of the area can be easily achieved:

$$\fint_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2} = \fint_{\mathbb{S}^2} e^{2u}\mathrm{d}\mu_{\mathbb{S}^2} \ge \frac{1}{\max f}.$$

Take $\epsilon=1/6$ in Proposition 4.1 and apply Lemma 2.1 to conclude that

$$\begin{aligned} E[u_0] \ge E[u] = E[v] &= \frac{1}{3}\fint_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} + 2\fint_{\mathbb{S}^2} v\mathrm{d}\mu_{\mathbb{S}^2} + \frac{2}{3}\fint_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} \\ &\ge -\log(C_{\frac{1}{6}}\max f) + \frac{1}{3}\fint_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}, \end{aligned}$$

which provides us the first estimate

$$\int_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} \le E[u_0] + \log(C_{\frac{1}{6}}\max f). \tag{4.2}$$

A simple observation, by Lemma 2.1, gives an upper bound on the average of the function $v$

$$\bar{v} := \fint_{\mathbb{S}^2} v\mathrm{d}\mu_{\mathbb{S}^2} \le \frac{1}{2}E[v] = \frac{1}{2}E[u] \le \frac{1}{2}E[u_0].$$

The lower bound of $\bar{v}$ needs the help of Aubin's inequality. Namely, set $\epsilon=1/4$ in Proposition 4.1 to obtain

$$
\begin{aligned}
-\log(\max f) \leq \log \fint_{\mathbb{S}^2} e^{2v} \mathrm{d}\mu_{\mathbb{S}^2} &\leq \log C_{\frac{1}{4}} + \frac{3}{4} \fint_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2 \fint_{\mathbb{S}^2} v \mathrm{d}\mu_{\mathbb{S}^2} \\
&= \log C_{\frac{1}{4}} + \frac{3}{4} E[v] + \frac{1}{2} \bar{v} \\
&\leq \log C_{\frac{1}{4}} + \frac{3}{4} E[u_0] + \frac{1}{2} \bar{v}
\end{aligned}
$$

with a positive constant $C$ depending on $\max f$, $E[u_0]$, together with the upper bound as above, we hence obtain

$$
|\bar{v}| \leq C. \tag{4.3}
$$

With the bounds (4.3) and (4.2) at hand, Onofri-Moser-Trudinger's inequality can be applied to show that, for any $p \geq 1$, there holds

$$
\sup_t \int_{\mathbb{S}^2} e^{2p|v|} \mathrm{d}\mu_{\mathbb{S}^2} \leq C(p).
$$

Now we carry over the argument to $u$ to see the similar estimate.

**Lemma 4.2.** *For any $T > 0$, any smooth solution of the flow* (2.2) *satisfies*

$$
\sup_{0 \leq t < T} \int_{\mathbb{S}^2} e^{4|u|} \mathrm{d}\mu_{\mathbb{S}^2} \leq C(T).
$$

**Proof.** Consider the normalized function

$$
v(t) = u \circ \varphi + \frac{1}{2} \log \det \mathrm{d}\varphi.
$$

Notice that Eq. (4.1) can also be written as, with $y = \varphi(x)$,

$$
0 = \int_{\mathbb{S}^2} x e^{2v(x)} \mathrm{d}\mu_{\mathbb{S}^2}(x) = \int_{\mathbb{S}^2} \varphi^{-1}(y) e^{2u(y)} \mathrm{d}\mu_{\mathbb{S}^2}(y),
$$

where

$$
\varphi(x) := \varphi_{P(t),\lambda(t)}(x) = \frac{2\lambda(x - (P \cdot x)P) + (\lambda^2(1 + P \cdot x) - 1 + P \cdot x)P}{\lambda^2(1 + P \cdot x) + 1 - P \cdot x},
$$

with $P(t) \in \mathbb{S}^2$, $\lambda(t) \geq 1$, and a well-known fact:

$$
\varphi^{-1}(y) = \varphi_{-P,\lambda}(y) = \frac{2\lambda(y - (P \cdot y)P) - (\lambda^2(1 - P \cdot y) - 1 - P \cdot y)P}{\lambda^2(1 - P \cdot y) + 1 + P \cdot y} =: \frac{A(y)}{B(y)}.
$$

Take the derivative with respect to the time $t$ to achieve

$$0=\int\limits_{\mathbb{S}^2}\frac{\partial}{\partial t}(\varphi^{-1}(y))e^{2u(y)}\mathrm{d}\mu_{\mathbb{S}^2}(y)+2\int\limits_{\mathbb{S}^2}\varphi^{-1}(y)e^{2u(y)}(\alpha f-K)(y)\mathrm{d}\mu_{\mathbb{S}^2}(y).$$

One calculates the first term first.

$$\begin{aligned}&\int\limits_{\mathbb{S}^2}\frac{\partial}{\partial t}(\varphi^{-1}(y))e^{2u(y)}\mathrm{d}\mu_{\mathbb{S}^2}(y)\\ =&\int\limits_{\mathbb{S}^2}\left(\frac{A_t(y)}{B(y)}-\frac{A(y)}{B^2(y)}B_t(y)\right)e^{2u(y)}\mathrm{d}\mu_{\mathbb{S}^2}(y)\\ =&\int\limits_{\mathbb{S}^2}\left(\frac{A_t\circ\varphi(x)}{B\circ\varphi(x)}-x\frac{B_t\circ\varphi(x)}{B\circ\varphi(x)}\right)e^{2v(x)}\mathrm{d}\mu_{\mathbb{S}^2}(x),\end{aligned}$$

where, in the last step, we have used the change of variables.

Recall definition of $A$ to have

$$\begin{aligned}A_t\circ\varphi(x)=&2\lambda_t(\varphi-P\cdot\varphi(x)P)-2\lambda[(\frac{\mathrm{d}P}{\mathrm{d}t}\cdot\varphi(x))P+(P\cdot\varphi(x))\frac{\mathrm{d}P}{\mathrm{d}t}]\\ &-[2\lambda\lambda_t(1-P\cdot\varphi(x))-(\lambda^2+1)\frac{\mathrm{d}P}{\mathrm{d}t}\cdot\varphi(x)]P\\ &-[\lambda^2(1-P\cdot\varphi)-1-P\cdot\varphi(x)]\frac{\mathrm{d}P}{\mathrm{d}t}.\end{aligned}$$

Similar calculation for $B$ provides

$$B_t\circ\varphi(x)=2\lambda\lambda_t(1-P\cdot\varphi(x))+(1-\lambda^2)\frac{\mathrm{d}P}{\mathrm{d}t}\cdot\varphi(x).$$

To proceed, by direct calculation, we have

$$P\cdot\varphi(x)=\frac{\lambda^2(1+P\cdot x)-1+P\cdot x}{\lambda^2(1+P\cdot x)+1-P\cdot x}. \tag{4.4}$$

This immediately shows the useful identity

$$[B\circ\varphi](x)=\frac{4\lambda^2}{\lambda^2(1+P\cdot x)+1-P\cdot x}. \tag{4.5}$$

Notice $P\cdot P=1$ for all $t\geq 0$ implies that $\frac{\mathrm{d}P}{\mathrm{d}t}\cdot P=0$. As a consequence, we obtain

$$\frac{\mathrm{d}P}{\mathrm{d}t}\cdot\varphi(x)=\frac{2\lambda\frac{\mathrm{d}P}{\mathrm{d}t}\cdot x}{\lambda^2(1+P\cdot x)+1-P\cdot x}$$

The following identity can be directly checked

$$\varphi(x)-(P\cdot\varphi(x))P=\frac{2\lambda(x-(P\cdot x)P)}{\lambda^2(1+P\cdot x)+(1-P\cdot x)}. \tag{4.6}$$

Similar calculation displays

$$\lambda^2(1-P\cdot\varphi(x))-(1+P\cdot\varphi(x))=\frac{-4\lambda^2(P\cdot x)}{\lambda^2(1+P\cdot x)+(1-P\cdot x)}. \tag{4.7}$$

Now combine above four labeled equations to have

$$\frac{A_t\circ\varphi}{B\circ\varphi}(x)=\frac{\lambda_t}{\lambda}(x-P)+\frac{(\lambda-1)^2}{2\lambda}[(\frac{\mathrm{d}P}{\mathrm{d}t}\cdot x)P-(P\cdot x)\frac{\mathrm{d}P}{\mathrm{d}t}]-\frac{\lambda^2-1}{2\lambda}\frac{\mathrm{d}P}{\mathrm{d}t}.$$

Notice that $\int_{\mathbb{S}^2} xe^{2v}\mathrm{d}\mu_{\mathbb{S}^2}=0$. Therefore, upon an integration, we have

$$\int_{\mathbb{S}^2}\frac{A_t\circ\varphi}{B\circ\varphi}(x)e^{2v(x)}\mathrm{d}\mu_{\mathbb{S}^2}(x)=-\frac{\lambda_t}{\lambda}(\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})P-\frac{\lambda^2-1}{2\lambda}(\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})\frac{\mathrm{d}P}{\mathrm{d}t}.$$

Term for $B$ can be treated in a similar manner to obtain

$$\frac{B_t\circ\varphi}{B\circ\varphi}(x)=\frac{\lambda_t}{\lambda}(1-P\cdot x)+\frac{1-\lambda^2}{2\lambda}\frac{\mathrm{d}P}{\mathrm{d}t}\cdot x.$$

Integrate it to get

$$\int_{\mathbb{S}^2}x\frac{B_t\circ\varphi}{B\circ\varphi}(x)e^{2v(x)}\mathrm{d}\mu_{\mathbb{S}^2}(x)=-\frac{\lambda_t}{\lambda}\int_{\mathbb{S}^2}x(P\cdot x)e^{2v}\mathrm{d}\mu_{\mathbb{S}^2}+\frac{1-\lambda^2}{2\lambda}\int_{\mathbb{S}^2}x(\frac{\mathrm{d}P}{\mathrm{d}t}\cdot x)e^{2v}\mu_{\mathbb{S}^2}.$$

For later reference, we write above calculation as

$$(\frac{\partial}{\partial t}\varphi^{-1})\circ\varphi(x)=\left[\frac{1}{\lambda}(\frac{\mathrm{d}\lambda}{\mathrm{d}t})P\wedge x+\frac{(\lambda-1)^2}{2\lambda}(\frac{\mathrm{d}P}{\mathrm{d}t})\wedge P+\frac{\lambda^2-1}{2\lambda}(\frac{\mathrm{d}P}{\mathrm{d}t})\wedge x\right]\wedge x, \tag{4.8}$$

where $\wedge$ is just usual cross product.

Let us denote the matrix $(\int_{\mathbb{S}^2} x^ix^je^{2v}\mathrm{d}\mu_{\mathbb{S}^2})$ by $\Lambda$. Above calculations show

$$\begin{aligned}&\frac{\lambda_t}{\lambda}\left(\Lambda-(\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2}\right)I)P+\frac{\lambda^2-1}{2\lambda}\left(\Lambda-(\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})I\right)\frac{\mathrm{d}P}{\mathrm{d}t}\\&=-2\int_{\mathbb{S}^2}\varphi^{-1}(x)e^{2u}(\alpha f-K)\mathrm{d}\mu_{\mathbb{S}^2}.\end{aligned}$$

Now we claim that the matrix $\Lambda - (\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})I$ is negative definite. To this end, for any unit vector $\vec{C} = (C^1, C^2, C^3)$, consider

$$\begin{aligned}
&\langle((\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})I - \Lambda)\vec{C}, \vec{C}\rangle \\
&= \int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2} - \int_{\mathbb{S}^2} e^{2v}(x\cdot\vec{C})^2\mathrm{d}\mu_{\mathbb{S}^2} \\
&= \int_{\mathbb{S}^2} e^{2v}(1-(x\cdot\vec{C})^2)\mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned}$$

We need to show this is bounded from below by a positive constant. Here we adopt similar argument as in the proof of [10, Lemma 4.1].

Set $\gamma = x\cdot\vec{C}$. Note that $1-\gamma^2 \ge 0$. Thus the Hölder's inequality indicates that

$$\begin{aligned}
\int_{\mathbb{S}^2} e^{2v}(1-\gamma^2)\mathrm{d}\mu_{\mathbb{S}^2} &\ge (\int_{\mathbb{S}^2}(1-\gamma^2)\mathrm{d}\mu_{\mathbb{S}^2})^2(\int_{\mathbb{S}^2} e^{-2v}(1-\gamma^2)\mathrm{d}\mu_{\mathbb{S}^2})^{-1} \\
&\ge (\int_{\mathbb{S}^2}(1-\gamma^2)\mathrm{d}\mu_{\mathbb{S}^2})^2(\int_{\mathbb{S}^2} e^{-4v}\mathrm{d}\mu_{\mathbb{S}^2})^{-\frac{1}{2}}(\int_{\mathbb{S}^2}(1-\gamma^2)^2\mathrm{d}\mu_{\mathbb{S}^2})^{-\frac{1}{2}} \\
&\ge (\frac{8\pi}{3})^{\frac{3}{2}}(\int_{\mathbb{S}^2} e^{-4v}\mathrm{d}\mu_{\mathbb{S}^2})^{-\frac{1}{2}},
\end{aligned}$$

where we have used the fact that $0 \le 1-\gamma^2 \le 1$.

Thus the Onofri-Moser-Trudinger inequality can be applied to get control on the integral $(\int_{\mathbb{S}^2} e^{-4v}\mathrm{d}\mu_{\mathbb{S}^2})$, namely,

$$(\int_{\mathbb{S}^2} e^{-4v}\mathrm{d}\mu_{\mathbb{S}^2}) \le 4\pi e^{4E[v]+12|\bar{v}|} \le C(T).$$

This is enough to ensure our claim. Hence the claim guarantees that the matrix $(\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})I - \Lambda$ has bounded inverse. Use this to rewrite our constraint identity as follows

$$\frac{\lambda_t}{\lambda}P + \frac{\lambda^2-1}{2\lambda}\frac{\mathrm{d}P}{\mathrm{d}t} = 2\left((\int_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})I - \Lambda\right)^{-1}\int_{\mathbb{S}^2}\varphi^{-1}e^{2u}(\alpha f - K)\mathrm{d}\mu_{\mathbb{S}^2}.$$

Take the norm on both sides, together with the facts that $P\cdot\frac{\mathrm{d}P}{\mathrm{d}t} = 0$, $P\cdot P = 1$ and $|\varphi^{-1}| = 1$, to conclude

$$\begin{aligned}
&\left(\frac{\lambda_t}{\lambda}\right)^2+\frac{\lambda^2-1}{2\lambda}\left|\frac{\mathrm{d}P}{\mathrm{d}t}\right|^2 \\
\leq& C(T,u_0)(\int_{\mathbb{S}^2} e^{2u}|\alpha f-K|\mathrm{d}\mu_{\mathbb{S}^2})^2 \\
\leq& C(T,u_0)(\int_{\mathbb{S}^2} e^{2u}\mathrm{d}\mu_{\mathbb{S}^2})\int_{\mathbb{S}^2}(\alpha f-K)^2 e^{2u}\mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned} \tag{4.9}$$

With this at hand, as well as (2.8), the estimate on the dilation factor $\lambda$ can be easily obtained, for any $0\leq t\leq T$,

$$\begin{aligned}
\left|\log\frac{\lambda(t)}{\lambda(0)}\right| \leq& \int_0^t\left|\frac{\lambda_\tau}{\lambda}\right|\mathrm{d}\tau \\
&\leq C(T,f,u_0)\int_0^t(\int_{\mathbb{S}^2} e^{2u}(\alpha f-K)^2\mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}\mathrm{d}\tau \\
&\leq C(T,f,u_0)\sqrt{t}(\int_0^t\int_{\mathbb{S}^2} e^{2u}(\alpha f-K)^2\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}\tau)^{\frac{1}{2}} \\
&\leq C(T,f,u_0).
\end{aligned}$$

Denote $\beta=\log\det\mathrm{d}\varphi$, we have $v=u\circ\varphi+\frac{1}{2}\beta$. Direct calculation shows

$$\int_{\mathbb{S}^2} e^{4u}\mathrm{d}\mu_{\mathbb{S}^2}=\int_{\mathbb{S}^2} e^{4u\circ\varphi}\det\mathrm{d}\varphi\mathrm{d}\mu_{\mathbb{S}^2}=\int_{\mathbb{S}^2} e^{4v}e^{-\beta}\mathrm{d}\mu_{\mathbb{S}^2},$$

and

$$\int_{\mathbb{S}^2} e^{-4u}\mathrm{d}\mu_{\mathbb{S}^2}=\int_{\mathbb{S}^2} e^{-4v}e^{3\beta}\mathrm{d}\mu_{\mathbb{S}^2}.$$

Recall the well-known fact that

$$\det\mathrm{d}\varphi=\left(\frac{2\lambda}{\lambda^2(1+P\cdot x)+1-P\cdot x}\right)^2,$$

which implies that

$$\beta=2\log\frac{2\lambda}{\lambda^2(1+P\cdot x)+1-P\cdot x}.$$

Notice, since $\lambda \geq 1$,

$$2 \leq \lambda^2(1 + P \cdot x) + 1 - P \cdot x \leq 2\lambda^2.$$

Thus there exists a constant $C(T, f, u_0) > 0$ such that $|\beta| \leq C(T, f, u_0)$ for all $0 \leq t \leq T$.

The lemma follows from this easily. $\square$

The key step to develop the theory of the global existence is the following $H^2(\mathbb{S}^2)$ bound for $u$.

**Lemma 4.3.** *For any $T > 0$, there exists a positive constant $C(T) > 0$ such that the following estimate holds:*

$$\sup_{0 \leq t < T} \|u(t)\|^2_{H^2(\mathbb{S}^2)} \leq C(T).$$

**Proof.** As a direct consequence of Lemma 4.2, $\|u\|_{H^1(\mathbb{S}^2)}$ is bounded in $[0, T]$. To that end, by Jensen's inequality we have

$$|\bar{u}| \leq \frac{1}{2} \log \fint_{\mathbb{S}^2} e^{2|u|} \mathrm{d}\mu_{\mathbb{S}^2} \leq C(T)$$

and

$$\fint_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} = E[u] - 2\bar{u} \leq E[u_0] + \log \fint_{\mathbb{S}^2} e^{2|u|} \mathrm{d}\mu_{\mathbb{S}^2} \leq C(T),$$

$$\|u\|_{L^2(\mathbb{S}^2)} \leq \|u - \bar{u}\|_{L^2(\mathbb{S}^2)} + \sqrt{4\pi}|\bar{u}| \leq \frac{1}{\sqrt{2}} \|\nabla u\|_{L^2(\mathbb{S}^2)} + \sqrt{4\pi}|\bar{u}| \leq C(T)$$

for all $0 \leq t \leq T$.

Now we consider the second order derivative estimate. We closely follow the Brendle's argument in [3]. In order to do so, let us set $w = (e^u)_t = e^u u_t$ and in other words, $u_t = e^{-u} w$. First let us rewrite the flow equation (2.2) as

$$\Delta_{\mathbb{S}^2} u = e^u w - \alpha(t) f e^{2u} + 1.$$

Then a direct computation shows

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} u)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
=&2 \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} u) \Delta_{\mathbb{S}^2} (e^{-u} w) \mathrm{d}\mu_{\mathbb{S}^2} \\
=&-2 \int_{\mathbb{S}^2} \langle \nabla (e^u w - \alpha(t) f e^{2u} + 1), \nabla (e^{-u} w) \rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}
\end{aligned}$$

$$
\begin{aligned}
&= -2\int_{\mathbb{S}^2} \langle \nabla(e^u w), \nabla(e^{-u}w)\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2\alpha(t)\int_{\mathbb{S}^2} \langle \nabla(fe^{2u}), \nabla(e^{-u}w)\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \\
&= -[\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} - 2\int_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2}] \\
&\quad - [\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} - 2\alpha(t)\int_{\mathbb{S}^2} \langle \nabla(fe^{2u}), \nabla(e^{-u}w)\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}] \\
&= I + II.
\end{aligned}
$$

Rewrite the first term as follows:

$$
\begin{aligned}
I &= -[\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} - 2\int_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2}] \\
&= -\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2\int_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
&\le -\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2[\int_{\mathbb{S}^2} |\nabla u|^4_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}]^{\frac{1}{2}} [\int_{\mathbb{S}^2} w^4 \mathrm{d}\mu_{\mathbb{S}^2}]^{\frac{1}{2}} \\
&\le -\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2C(T)\left[ (\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} u)^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + 1][\|w\|_{L^2(\mathbb{S}^2)} \|\nabla w\|_{L^2(\mathbb{S}^2)} + \|w\|^2_{L^2(\mathbb{S}^2)} \right] \\
&\le C(T)[\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} u)^2 \mathrm{d}\mu_{\mathbb{S}^2}) + 1][\|w\|^2_{L^2(\mathbb{S}^2)} + 1].
\end{aligned}
$$

Second term on the right hand side can also be controlled in a similar fashion.

For convenience, in the followings, the positive constant $C(T)$ may be different from line to line.

Firstly we manage to give some useful estimates in terms of its $H^1(\mathbb{S}^2)$ norm and $L^4(\mathbb{S}^2)$ norm of $e^u$

$$
|\alpha| = |(\int_{\mathbb{S}^2} f^2 \mathrm{d}\mu_g)^{-1} (\int_{\mathbb{S}^2} (\langle \nabla f, \nabla u\rangle_{\mathbb{S}^2} + f) \mathrm{d}\mu_{\mathbb{S}^2})| \le C(T);
$$

and the routine estimate of $L^4(\mathbb{S}^2)$ norm of $|\nabla u|_{\mathbb{S}^2}$ in terms of $L^2(\mathbb{S}^2)$ norm of $\Delta_{\mathbb{S}^2} u$, i.e.

$$
\begin{aligned}
&\int_{\mathbb{S}^2} |\nabla u|^4_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \\
&\le C(\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} u)^2 \mathrm{d}\mu_{\mathbb{S}^2} \int_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + (\int_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^2)
\end{aligned}
$$

$$\leq C(T)(\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2 \mathrm{d}\mu_{\mathbb{S}^2}+1),$$

which has been used $H^1(\mathbb{S}^2)$ estimate of $u$.

We are ready to estimate the second term

$$\begin{aligned}
II =& -\int_{\mathbb{S}^2}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\\
&+2\alpha\int_{\mathbb{S}^2}e^u(-w\langle\nabla f,\nabla u\rangle_{\mathbb{S}^2}-2fw|\nabla u|^2_{\mathbb{S}^2}+\langle\nabla f,\nabla w\rangle_{\mathbb{S}^2}+2f\langle\nabla u,\nabla w\rangle_{\mathbb{S}^2})\mathrm{d}\mu_{\mathbb{S}^2}\\
\leq& -\int_{\mathbb{S}^2}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}+C_1(T)\int_{\mathbb{S}^2}e^u(|\nabla u|_{\mathbb{S}^2}|w|+|w||\nabla u|^2_{\mathbb{S}^2}+|\nabla w|_{\mathbb{S}^2}\\
&+|\nabla u|_{\mathbb{S}^2}\cdot|\nabla w|_{\mathbb{S}^2})\mathrm{d}\mu_{\mathbb{S}^2}\\
=& -\int_{\mathbb{S}^2}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}+II_1+II_2+II_3+II_4.
\end{aligned}$$

Now we estimate term-wise: first term $II_1$ can be controlled as follows,

$$\begin{aligned}
II_1 =& C_1(T)\int_{\mathbb{S}^2}|\nabla u|_{\mathbb{S}^2}e^u|w|\mathrm{d}\mu_{\mathbb{S}^2}\leq C(T)(\int_{\mathbb{S}^2}w^2e^{2u}\mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}\\
\leq& C(T)\|w\|_{L^4(\mathbb{S}^2)}\leq C(T)(\|w\|^2_{L^2(\mathbb{S}^2)}+1)+\frac{1}{4}\|\nabla w\|^2_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

Secondly, term $II_2$ is dominated by

$$\begin{aligned}
II_2 =& C_1(T)\int_{\mathbb{S}^2}|w|e^u|\nabla u|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\leq\|\nabla u\|^2_{L^4(\mathbb{S}^2)}\|we^u\|_{L^2(\mathbb{S}^2)}\\
\leq& C(T)(\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2}+1)^{\frac{1}{2}}\|w\|_{L^4(\mathbb{S}^2)}\\
\leq& C(T)(\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2}+1)^{\frac{1}{2}}(\|\nabla w\|_{L^2(\mathbb{S}^2)}+\|w\|_{L^2(\mathbb{S}^2)})\\
\leq& C(T)(\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2}+1)(\|w\|^2_{L^2(\mathbb{S}^2)}+1)+\frac{1}{4}\|\nabla w\|^2_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

Then the third term $II_3$ is relatively simple:

$$II_3=C_1(T)\int_{\mathbb{S}^2}e^u|\nabla w|_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\leq C(T)+\frac{1}{4}\|\nabla w\|^2_{L^2(\mathbb{S}^2)}.$$

Finally we treat term $II_4$:

$$\begin{aligned}
II_4 =& C_1(T)\int\limits_{\mathbb{S}^2} e^u |\nabla u|_{\mathbb{S}^2}\cdot|\nabla w|_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\\
\leq& C(T)\int\limits_{\mathbb{S}^2} e^{2u}|\nabla u|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}+\frac{1}{4}\|\nabla w\|^2_{L^2(\mathbb{S}^2)}\\
\leq& C(T)\|\nabla u\|^2_{L^4(\mathbb{S}^2)}+\frac{1}{4}\|\nabla w\|^2_{L^2(\mathbb{S}^2)}\\
\leq& C(T)(\int\limits_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2}+1)+\frac{1}{4}\|\nabla w\|^2_{L^2(\mathbb{S}^2)}
\end{aligned}$$

Summarize the above estimates from $II_1$ to $II_4$ to have

$$II\leq C(T)(\int\limits_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2}+1)(\|w\|^2_{L^2(\mathbb{S}^2)}+1).$$

Therefore combining the estimations of $I$ and $II$, we finally conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}(1+\int\limits_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2})\leq C(T)(\int\limits_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u)^2\mathrm{d}\mu_{\mathbb{S}^2}+1)(\int\limits_{\mathbb{S}^2}w^2\mathrm{d}\mu_{\mathbb{S}^2}+1).$$

Finally, for any $0\leq t<T$, after integration, we have

$$\int\limits_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u(t))^2\mathrm{d}\mu_{\mathbb{S}^2}+1\leq[\int\limits_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}u(0))^2\mathrm{d}\mu_{\mathbb{S}^2}+1)]\exp\left\{C(T)(\int\limits_0^T\int\limits_{\mathbb{S}^2}w^2\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}t+T)\right\}.$$

Then we have

$$\sup_{0\leq t<T}\|u\|_{H^2(\mathbb{S}^2)}\leq C(T),$$

by observing that $\int_{\mathbb{S}^2}w^2\mathrm{d}\mu_{\mathbb{S}^2}=\int_{\mathbb{S}^2}(\alpha f-K)^2\mathrm{d}\mu_{\mathbb{S}^2}$.

This completes the proof of Lemma 4.3. □

The last ingredient for global existence is the local Hölder estimate which follows from the previous lemma.

**Proposition 4.4** *(Simon Brendle [5, Proposition 2.6]). Given any $T>0$ and any $0<\mu<1$, there exists a constant $C(T)$ such that*

$$|u(x_1,t_1)-u(x_2,t_2)|\leq C(T)((t_1-t_2)^{\frac{\mu}{2}}+d(x_1,x_2)^{\mu})$$

*for all $x_1,x_2\in\mathbb{S}^2$ and all $t_1,t_2\in[0,T]$ satisfying $0<t_1-t_2<1$.*

**Proof.** For all $0 \le t \le T$, by Lemma 4.3 and Sobolev embedding, we have $u \in C^{\mu}(\mathbb{S}^2)$ for all $0 \le \mu < 1$. Fix a number $0 < \mu < 1$,

$$|u(x_1, t) - u(x_2, t)| \le C(T)d(x_1, x_2)^{\mu}$$

for all $x_1, x_2 \in \mathbb{S}^2$ and $t \in [0, T]$. By Lemma 4.3 and (2.3), it is easy to show

$$\int_{\mathbb{S}^2} (Ke^{2u})^2 \mathrm{d}\mu_{\mathbb{S}^2} = \int_{\mathbb{S}^2} (-\Delta_{\mathbb{S}^2} u + 1)^2 \mathrm{d}\mu_{\mathbb{S}^2} \le C(T).$$

It follows from this that

$$\int_{\mathbb{S}^2} K^2 \mathrm{d}\mu_{\mathbb{S}^2} \leqslant C(T),$$

as well as

$$|\alpha| \leqslant (\int_{\mathbb{S}^2} f^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2})^{-1} |\int_{\mathbb{S}^2} f(\Delta_{\mathbb{S}^2} u + 1) \mathrm{d}\mu_{\mathbb{S}^2}| \leqslant C(T),$$

whence

$$\int_{\mathbb{S}^2} u_t^2 \mathrm{d}\mu_{\mathbb{S}^2} = \int_{\mathbb{S}^2} (\alpha f - K)^2 \mathrm{d}\mu_{\mathbb{S}^2} \le C(T).$$

With this at hand, it is standard to show

$$\begin{aligned}
|u(x, t_1) - u(x, t_2)| &\le C(t_1 - t_2)^{-1} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(x, t_1) - u(x, t_2)| \mathrm{d}\mu_{\mathbb{S}^2}(y) \\
&\le C(t_1 - t_2)^{-1} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u(t_1)(y) - u(t_2)(y)| \mathrm{d}\mu_{\mathbb{S}^2} + C(T)(t_1 - t_2)^{\frac{\mu}{2}} \\
&\le C \sup_{t_2 \le t \le t_1} \int_{B_{\sqrt{t_1 - t_2}}(x)} |u_t| \mathrm{d}\mu_{\mathbb{S}^2} + C(T)(t_1 - t_2)^{\frac{\mu}{2}} \\
&\le C(t_1 - t_2)^{\frac{1}{2}} (\sup_{t_2 \le t \le t_1} \int_{\mathbb{S}^2} u_t^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + C(T)(t_1 - t_2)^{\frac{\mu}{2}} \\
&\le C(T)(t_1 - t_2)^{\frac{\mu}{2}}
\end{aligned}$$

for all $x \in \mathbb{S}^2$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$. Notice that, in the last step, we have used the fact that $\frac{1}{2} > \frac{\mu}{2}$ and $0 < t_2 - t_1 < 1$. This proves the assertions. $\square$

With this Hölder estimate, the standard regularity theory for parabolic equations can be applied to bound all higher order derivatives of the function $u$ on every finite time interval $[0, T]$, hence the flow globally exists in time.

## 5. $L^p$ and $H^1$ convergences

Let us define two functions: $F_p(t) := \fint_{\mathbb{S}^2} |\alpha f - K|^p \mathrm{d}\mu_g$ for any $p \geq 1$ and $G_2(t) := \fint_{\mathbb{S}^2} |\nabla(\alpha f - K)|_g^2 \mathrm{d}\mu_g$. The main purpose of this section is to show that both $F_p$ and $G_2$ tends to zero as the time $t$ goes to $\infty$.

As a first step, one starts with $F_2$.

**Lemma 5.1.** *There holds* $\lim_{t\to\infty} F_2(t) = 0$.

**Proof.** We first need an evolution equation of the Gaussian curvature $K$. By computation, it can be easily achieved:

$$K_t = \frac{\partial}{\partial t}(e^{-2u}(-\Delta_{\mathbb{S}^2} u + 1)) = 2K(K - \alpha f) + \Delta_g(K - \alpha f). \tag{5.1}$$

Keep in mind, the choice of $\alpha$ gives

$$\int_{\mathbb{S}^2} (K - \alpha f) f \mathrm{d}\mu_g = 0.$$

This will be used to handle the term with $\alpha_t$ in the following calculation

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2} (K - \alpha f)^2 \mathrm{d}\mu_g &= \int_{\mathbb{S}^2} (K - \alpha f)(K_t - \alpha_t f) - (K - \alpha f)^3 \mathrm{d}\mu_g \\
&= \int_{\mathbb{S}^2} K_t (K - \alpha f) \mathrm{d}\mu_g - \int_{\mathbb{S}^2} (K - \alpha f)^3 \mathrm{d}\mu_g \\
&= \int_{\mathbb{S}^2} (2K(K - \alpha f) + \Delta_g(K - \alpha f))(K - \alpha f) \mathrm{d}\mu_g - \int_{\mathbb{S}^2} (K - \alpha f)^3 \mathrm{d}\mu_g \\
&= -\int_{\mathbb{S}^2} \left[|\nabla(K - \alpha f)|_g^2 - 2\alpha f(\alpha f - K)^2\right] \mathrm{d}\mu_g + \int_{\mathbb{S}^2} (K - \alpha f)^3 \mathrm{d}\mu_g.
\end{aligned}$$

Recall our normalization again $\fint_{\mathbb{S}^2} f e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} = 1$ and topological invariant $\fint_{\mathbb{S}^2} K e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} = 1$, we have

$$\alpha - 1 = \fint_{\mathbb{S}^2} (\alpha f - K) e^{2u} \mathrm{d}\mu_{\mathbb{S}^2},$$

which implies the estimate

$$
\begin{aligned}
|\alpha-1|^2 &= \left(\fint_{\mathbb{S}^2}(\alpha f-K)e^{2u}\mathrm{d}\mu_{\mathbb{S}^2}\right)^2\\
&\le \fint_{\mathbb{S}^2}(\alpha f-K)^2\mathrm{d}\mu_g \fint_{\mathbb{S}^2} e^{2u}\mathrm{d}\mu_{\mathbb{S}^2}\\
&\le e^{E[u_0]}\int_{\mathbb{S}^2}(\alpha f-K)^2\mathrm{d}\mu_g.
\end{aligned}
$$

Refresh us the notations: $h=e^{2v}g_{\mathbb{S}^2}=\varphi^*g,\ K_h=K_g\circ\varphi$, and $f_\varphi=f\circ\varphi$. By the estimate (2.8), for any $0<\epsilon_0<1$, there exists a large $t_0$ such that

$$
\left.\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g\right|_{t=t_0}<\epsilon_0.
$$

Choose a maximal $t_1>t_0$ such that

$$
\sup_{t_0\le t<t_1}\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g\le 2\epsilon_0.
$$

It follows that, for any $t_0<t<t_1$,

$$
\|K_h\|_{L^2(\mathbb{S}^2,h)}=\|K\|_{L^2(\mathbb{S}^2,g)}\le\|K-\alpha f\|_{L^2(\mathbb{S}^2,g)}+|\alpha|\,\|f\|_{L^2(\mathbb{S}^2,g)}\le\sqrt{2\epsilon_0}+C.
$$

Our goal is to show that $t_1$ must be $+\infty$. To achieve this, first we recall, due to the conformal invariant property, the following equation for $v$ holds true:

$$
-\Delta_{\mathbb{S}^2}v+1=K_he^{2v}.
$$

Thus, the boundedness on $v$ in $W^{2,p}(\mathbb{S}^2)$ with $p=4/3$ was derived through the estimate

$$
\begin{aligned}
\int_{\mathbb{S}^2}|-\Delta_{\mathbb{S}^2}v+1|^p\mathrm{d}\mu_{\mathbb{S}^2} &= \int_{\mathbb{S}^2}|K_h|^pe^{2pv}\mathrm{d}\mu_{\mathbb{S}^2}\\
&=\int_{\mathbb{S}^2}|K_h|^pe^{2(p-1)v}\mathrm{d}\mu_h\\
&\le\left(\int_{\mathbb{S}^2}K_h^2\mathrm{d}\mu_h\right)^{\frac{p}{2}}\left(\int_{\mathbb{S}^2}e^{\frac{2p}{2-p}v}\mathrm{d}\mu_{\mathbb{S}^2}\right)^{1-\frac{p}{2}}\\
&\le C(\epsilon_0).
\end{aligned}
$$

As one has seen before that the average of $v$ is bounded in its absolute values. The $L^\infty(\mathbb{S}^2)$ bound on $v$ follows from this.

With this estimate at hand, one easily obtains

$$\int_{\mathbb{S}^2}(-\Delta_{\mathbb{S}^2}v+1)^2\mathrm{d}\mu_{\mathbb{S}^2}=\int_{\mathbb{S}^2}K_h^2e^{4v}\mathrm{d}\mu_{\mathbb{S}^2}\le e^{2\|v\|_\infty}\int_{\mathbb{S}^2}K_h^2e^{2v}\mathrm{d}\mu_{\mathbb{S}^2}\le C_1(\epsilon_0)),$$

which turns to imply that $v(t)\in H^2(\mathbb{S}^2)$. By Hölder's and Sobolev inequalities, the $F_3$ can be controlled in terms $G_2$ and $F_2$:

$$\begin{aligned}\int_{\mathbb{S}^2}|K_h-\alpha f_\varphi|^3\mathrm{d}\mu_h \le& C_1\left\|K_h-\alpha f_\varphi\right\|_{L^2(\mathbb{S}^2,h)}\left\|K_h-\alpha f_\varphi\right\|^2_{L^4(\mathbb{S}^2)} \\ \le& C_1\left\|K_h-\alpha f_\varphi\right\|_{L^2(\mathbb{S}^2,h)}\left\|K_h-\alpha f_\varphi\right\|^2_{H^1(\mathbb{S}^2)} \\ \le& C_0\left\|K_h-\alpha f_\varphi\right\|_{L^2(\mathbb{S}^2,h)}[\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|_h^2+(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_h].\end{aligned}\tag{5.2}$$

Notice that $C(\epsilon_0)$ is a monotone increasing function of $\epsilon_0$ as is $C_1(\epsilon_0)$. Thus $\epsilon_0>0$ can be so small such that $\sqrt{2\epsilon_0}C_0\le 1/2$. It follows that the next estimate can be achieved

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_h+\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|_h^2\mathrm{d}\mu_h \\ =&2\alpha\int_{\mathbb{S}^2}f(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_h+\int_{\mathbb{S}^2}|K_h-\alpha f_\varphi|^3\mathrm{d}\mu_h \\ \le&2|\alpha|(\max|f|)\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_h+\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_h+\frac{1}{2}\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|_h^2\mathrm{d}\mu_h \\ \le&C(f)\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_h+\frac{1}{2}\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|_h^2\mathrm{d}\mu_h.\end{aligned}\tag{5.3}$$

Integrate the above inequality over $[t_0,t]$ for any $t\in[t_0,t_1)$ to show

$$\begin{aligned}&\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g+\int_{t_0}^{t_1}\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|_h^2\mathrm{d}\mu_h\mathrm{d}\tau \\ \le&\left.\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g\right|_{t=t_0}+2C\int_{t_1}^{\infty}\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g\mathrm{d}t.\end{aligned}$$

If $t_0$ is chosen large enough such that

$$4C(f)\int_{t_0}^{\infty}\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g\mathrm{d}t\le\epsilon_0,$$

then one obtains, for any $t\in[t_0,t_1)$,

$$\int_{\mathbb{S}^2}(K-\alpha f)^2\mathrm{d}\mu_g\le\frac{3}{2}\epsilon_0.$$

This forces that $t_1=+\infty$ and hence our claim: $\lim_{t\to\infty}F_2(t)=0$. $\square$

As a byproduct, we also conclude that

$$\int_0^{\infty}\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|_h^2\mathrm{d}\mu_h\mathrm{d}\tau\le C<\infty. \tag{5.4}$$

Next, with the help of the estimate (5.4), we prove the following fact.

**Lemma 5.2.** *There holds* $\lim_{t\to\infty}G_2(t)=0$.

**Proof.** Let us start with a calculation

$$\begin{aligned}\frac{\mathrm{d}G_2(t)}{\mathrm{d}t}=&2\int_{\mathbb{S}^2}\langle\nabla K_t-\alpha_t\nabla f,\nabla(K-\alpha f)\rangle_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\\=&-4\int_{\mathbb{S}^2}(K-\alpha f)^2\Delta_{\mathbb{S}^2}(K-\alpha f)\mathrm{d}\mu_{\mathbb{S}^2}+4\alpha\int_{\mathbb{S}^2}f(\alpha f-K)\Delta_{\mathbb{S}^2}(K-\alpha f)\mathrm{d}\mu_{\mathbb{S}^2}\\&-2\int_{\mathbb{S}^2}e^{-2u}(\Delta_{\mathbb{S}^2}(K-\alpha f))^2\mathrm{d}\mu_{\mathbb{S}^2}-2\alpha_t\int_{\mathbb{S}^2}\langle\nabla f,\nabla(K-\alpha f)\rangle_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2},\end{aligned} \tag{5.5}$$

where of course we have used the flow equation (5.1) for the Gaussian curvature.

Now we need to estimate every term on the right hand side of Eq. (5.5). First, by Hölder's and Young's inequalities, one has

$$\left|\int_{\mathbb{S}^2}(K-\alpha f)^2\Delta_{\mathbb{S}^2}(K-\alpha f)\mathrm{d}\mu_{\mathbb{S}^2}\right|\le2\int_{\mathbb{S}^2}(K-\alpha f)^4\mathrm{d}\mu_g+\frac{1}{8}\int_{\mathbb{S}^2}e^{-2u}(\Delta_{\mathbb{S}^2}(K-\alpha f))^2\mathrm{d}\mu_{\mathbb{S}^2}.$$

Notice that, as we have seen before, Lemma 5.1 implies that there exists a uniform constant $C$ which may depend only on $f$ and $u_0$ such that the normalized conformal factor $v$ is bounded by $C$ in its absolute values. Thus we can do change of variables to get

$$
\begin{aligned}
&\int_{\mathbb{S}^2}(K-\alpha f)^4 \mathrm{d}\mu_g=\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^4 \mathrm{d}\mu_h\\
\leq& e^{2C}\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^4 \mathrm{d}\mu_{\mathbb{S}^2}\\
\leq& e^{2C}[K(2,1)\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_{\mathbb{S}^2}]^2\\
\leq&(2K(2,1)+1)^2e^{2C}(\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_{\mathbb{S}^2})[(\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}\\
&+(\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}]^2\\
\leq& C_1(G_2(t)+F_2(t)),
\end{aligned}
$$

where we have used the following facts

$$
\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2\mathrm{d}\mu_{\mathbb{S}^2}\leq e^{2C}\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^2e^{2v}\mathrm{d}\mu_{\mathbb{S}^2}=e^{2C}F_2,
$$

and

$$
\int_{\mathbb{S}^2}|\nabla(K_h-\alpha f_\varphi)|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}=G_2
$$

by conformal invariance of Dirichlet integral.

The second estimate is rather easy

$$
\begin{aligned}
&\left|4\alpha\int_{\mathbb{S}^2}f(\alpha f-K)\Delta_{\mathbb{S}^2}(K-\alpha f)\mathrm{d}\mu_{\mathbb{S}^2}\right|\\
\leq& 8|\alpha|^2(\max|f|)^2\int_{\mathbb{S}^2}(K-\alpha f)^2e^{2u}\mathrm{d}\mu_{\mathbb{S}^2}+\frac{1}{2}\int_{\mathbb{S}^2}e^{-2u}(\Delta_{\mathbb{S}^2}(K-\alpha f))^2\mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned}
$$

The last term is quite involved. One needs to compute $\alpha'(t)$ first: by definition of $\alpha$,

$$
\alpha_t\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_g+2\alpha\int_{\mathbb{S}^2}f^2(\alpha f-K)\mathrm{d}\mu_g=\int_{\mathbb{S}^2}f\Delta_{\mathbb{S}^2}(K-\alpha f)\mathrm{d}\mu_{\mathbb{S}^2}.
$$

Notice that, the boundedness on $\int_{\mathbb{S}^2} e^{2u} \mathrm{d}\mu_{\mathbb{S}^2}$ implies

$$\left| \int_{\mathbb{S}^2} f^2(\alpha f - K)\mathrm{d}\mu_g \right| \leq C(f)\sqrt{F_2},$$

and

$$\left| \int_{\mathbb{S}^2} f \Delta_{\mathbb{S}^2}(K - \alpha f)\mathrm{d}\mu_{\mathbb{S}^2} \right| \leq (\int_{\mathbb{S}^2} f^2 \mathrm{d}\mu_g)^{\frac{1}{2}} (\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2}(K - \alpha f))^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}.$$

Combine these calculations to obtain the control on the last term

$$\begin{aligned}
&\left| 2\alpha_t \int_{\mathbb{S}^2} \langle \nabla f, \nabla (K - \alpha f)\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \right| \\
\leq & C(\sqrt{F_2} + (\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2}(K - \alpha f))^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}})(\int_{\mathbb{S}^2} |\nabla f|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} (\int_{\mathbb{S}^2} |\nabla (K - \alpha f)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} \\
\leq & C_1\sqrt{F_2}\sqrt{G_2} + C_2 G_2 + \int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2}(K - \alpha f))^2 \mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned}$$

Collect those three estimates to get

$$\frac{\mathrm{d}}{\mathrm{d}t}(G_2(t)) \leq C(F_2 + G_2),$$

with some positive constant $C$ independent of $t$.

Finally for any $0 < \epsilon < 1$, by estimate (5.4) and the integrability of the function $F_2$, there exists a $t_0 > 0$ such that $G_2(t_0) < \epsilon$ as well as

$$C \int_{t_0}^{\infty} (F_2 + G_2)\mathrm{d}t \leq \frac{1}{2}\epsilon.$$

Thus, up to an integration, for any $t > t_0$, one has

$$G_2(t) \leq G_2(t_0) + C \int_{t_0}^{\infty} (G_2 + F_2)\mathrm{d}t \leq 2\epsilon.$$

This yields $\lim_{t\to\infty} G_2 = 0$ as desired. □

Lemmas 5.1 and 5.2 make us easy to conclude that

**Lemma 5.3.** *For all $p > 1$ there holds $\lim_{t\to\infty} F_p(t) = 0$.*

**Proof.** Let $k \geq 2$ be an integer. First if $k = 2$, we have

$$\begin{aligned}
&[\int_{\mathbb{S}^2} (K - \alpha f)^{2k} \mathrm{d}\mu_g]^{\frac{1}{2}} = [\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^4 \mathrm{d}\mu_h]^{\frac{1}{2}} \\
\leq & e^C [\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^4 \mathrm{d}\mu_{\mathbb{S}^2}]^{\frac{1}{2}} \\
\leq & e^C [K(2,1) \int_{\mathbb{S}^2} |\nabla (K_h - \alpha f_\varphi)^2|_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^2 \mathrm{d}\mu_{\mathbb{S}^2}] \\
\leq & (2K(2,1) + 1) e^C (\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} \\
& \times [(\int_{\mathbb{S}^2} |\nabla (K_h - \alpha f_\varphi)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + (\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}] \\
\leq & C_1 (G_2(t) + F_2(t)).
\end{aligned}$$

Since the right hand side has its limit 0 as $t \to \infty$, we have $\lim_{t\to\infty} F_{2k}(t) = 0$ for $k = 2$. Now for any integer $k > 2$,

$$\begin{aligned}
&[\int_{\mathbb{S}^2} (K - \alpha f)^{2k} \mathrm{d}\mu_g]^{\frac{1}{2}} = [\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^{2k} \mathrm{d}\mu_h]^{\frac{1}{2}} \\
\leq & e^C [\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^{2k} \mathrm{d}\mu_{\mathbb{S}^2}]^{\frac{1}{2}} \\
\leq & e^C [K(2,1) \int_{\mathbb{S}^2} |\nabla (K_h - \alpha f_\varphi)^k|_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^k \mathrm{d}\mu_{\mathbb{S}^2}] \\
\leq & (kK(2,1) + 1) e^{2C} (\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^{2(k-1)} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} \\
& \times [(\int_{\mathbb{S}^2} |\nabla (K_h - \alpha f_\varphi)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + (\int_{\mathbb{S}^2} (K_h - \alpha f_\varphi)^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}}]^2 \\
\leq & C_1 (G_2(t) + F_2(t) + F_{2(k-1)}(t)).
\end{aligned}$$

Thus mathematical induction on $k$ implies that for all $k \geq 2$, we have $\lim_{t\to\infty} F_{2k}(t) = 0$. The rest of argument follows from the Hölder's inequality. $\square$

## 6. Blow-up analysis

From now on, we should always assume that our function $f$ on $\mathbb{S}^2$ cannot be realized as Gaussian curvature for some conformal metric. Under this assumption, along the flow, we must have

**Lemma 6.1.** *If $f$ cannot be realized as Gaussian curvature for some conformal metric, then we must have*

*(a)* $\lim_{t\to\infty} \|u\|_{H^2(\mathbb{S}^2)} = +\infty$;
*(b)* $\lim_{t\to\infty} \fint_{\mathbb{S}^2} u \mathrm{d}\mu_{\mathbb{S}^2} = -\infty$;
*(c)* $\lim_{t\to\infty} \max_{\mathbb{S}^2} u = +\infty$.

**Proof.** In fact, if there exists a sequence $\{t_k\}$ of the time $t$ such that one of three cases does not hold, then we can easily see that $f$ can be realized as a Gaussian curvature, as the corresponding sequence $\{u(t_k)\}$ converges to a limit function $u_\infty$ which solves Eq. (2.3) with Gaussian curvature equal to $f$. □

Thus now we consider the normalized conformal factor $v = u \circ \varphi + \frac{1}{2}\log(\det \mathrm{d}\varphi)$ where, as we have seen before, $\int_{\mathbb{S}^2} x e^{2v} \mathrm{d}\mu_{\mathbb{S}^2} = 0$ with $\varphi(x) = \varphi_{P(t),\lambda(t)}(x)$. Observe that $|P(t)| = 1$ and $\lambda(t) \geq 1$. Lemma 6.1 implies that $\lim_{t\to\infty} \lambda(t) = \infty$.

Then, with existed estimates we have set up in previous section, we summary it in the following lemma.

**Lemma 6.2.** *For any $0 < \gamma < 1$, we have*

$$v \in C^{1,\gamma}(\mathbb{S}^2).$$

**Proof.** By using Lemmas 5.1 and 5.2, we have

$$K_h - \alpha f_\varphi \in H^1(\mathbb{S}^2).$$

Making use of Sobolev inequality, for any $p > 2$,

$$K_h - \alpha f_\varphi \in L^p(\mathbb{S}^2).$$

Thus $K_h \in L^p(\mathbb{S}^2)$. Recall

$$-\Delta_{\mathbb{S}^2} v + 1 = K_h e^{2v}$$

Then $\Delta_{\mathbb{S}^2} v \in L^p(\mathbb{S}^2)$. Since $v \in L^\infty(\mathbb{S}^2)$, we have

$$v \in W^{2,p}(\mathbb{S}^2)$$

Let $1 + \gamma < 2 - \frac{2}{p}$ and use Sobolev embedding theorem to show

$$v \in C^{1,\gamma}(\mathbb{S}^2).$$

This completes the proof. □

Notice that our Lemma 6.1 precisely implies the following claim.

**Lemma 6.3.** *There exists at least one point $P_1 \in \mathbb{S}^2$ such that, for any $R > 0$,*

$$\liminf_{t\to\infty} \int\limits_{B_R(P_1)} |K_{g(t)}| \mathrm{d}\mu_{g(t)} \geq 2\pi,$$

*where $B_R(P_1)$ is a geodesic ball with center $P_1$ and radius $R$.*

**Proof.** We prove it by contradiction. Suppose that, for any $x_0 \in \mathbb{S}^2$, there exists some $R > 0$ such that

$$\limsup_{t\to\infty} \int\limits_{B_R(x_0)} |K(t)| \mathrm{d}\mu_{g(t)} \leq a < 2\pi.$$

Let $\omega(t) = u(t) \circ \pi^{-1} + \log \frac{2}{1+|z|^2}$. It is well-known that $\omega(t)$ satisfies

$$-\Delta_{\mathbb{R}^2}\omega(t) = (K_{g(t)} \circ \pi^{-1})e^{2\omega(t)} := f(t).$$

Up to some scaling, we assume that the image of $B_R(x_0)$ under the stereographic projection $\pi$ is the unit disk $B \subset \mathbb{R}^2$. Thus, we have

$$\|f(t)\|_{L^1(B)} = \int\limits_{B_R(x_0)} |K_{g(t)}| \mathrm{d}\mu_{g(t)} \leq a < 2\pi.$$

Splitting $\omega(t) = \omega^0(t) + \hat{\omega}(t)$, where $\omega^0(t)$ is harmonic in $B$ and $\hat{\omega}(t) = 0$ on $\partial B$. By using [4, Theorem 1] we obtain a uniform bound of

$$\int\limits_B e^{p|\hat{\omega}(t)|} \mathrm{d}x \leq C(p, a)$$

for any $p < 4\pi/a$. Since $a < 2\pi$, we may fix some such number $p > 2$. We remark here that the choice of $p$ is independent of $x_0$, rather only depending on $a$.

On the other hand, by the mean value property of harmonic functions, for any $y \in B_{1/2}(0) \subset \mathbb{R}^2$ we may estimate

$$\omega^0(t)(y) = \fint\limits_{B_{\frac{1}{2}}(y)} \omega^0(t)\mathrm{d}x \leq \fint\limits_{B_{\frac{1}{2}}(y)} \omega(t)\mathrm{d}x + \fint\limits_{B_{\frac{1}{2}}(y)} |\hat{\omega}(t)|\mathrm{d}x.$$

By Jensen's inequality and the boundedness of area, two terms on the right hand side can be controlled as:

$$2 \fint_{B_{\frac{1}{2}}(y)} \omega(t)\mathrm{d}x \le \log \fint_{B_{\frac{1}{2}}(y)} e^{2\omega(t)}\mathrm{d}x) \le C$$

and

$$\fint_{B_{\frac{1}{2}}(y)} |\hat{\omega}(t)|\mathrm{d}x \le \log \fint_{B_{\frac{1}{2}}(y)} e^{|\hat{\omega}(t)|}\mathrm{d}x \le C,$$

uniformly for $y \in B_{\frac{1}{2}}(0)$. Thus $e^{\omega(t)} \le Ce^{\hat{\omega}(t)}$ is bounded in $L^p(B_{\frac{1}{2}}(0))$. By Hölder's inequality, we have

$$\left\| K_{g(t)} \circ \pi^{-1} e^{\omega(t)} \right\|^2_{L^2(B)} = \int_{B_R(x_0)} |K(t)|^2 \mathrm{d}\mu_{g(t)} \le C.$$

Therefore we find that

$$-\Delta_{\mathbb{R}^2}\omega(t) = K_{g(t)} \circ \pi^{-1} e^{\omega(t)} \cdot e^{\omega(t)}$$

is bounded in $L^q(B_{1/2}(0))$ for some $q > 1$. Upon covering $\mathbb{S}^2$ with finitely many geodesic balls $B_{R_i}(x_i)$ corresponding to the ball $B_{1/2}(0)$, under the stereo-graphic projection, we obtain that $\Delta_{\mathbb{S}^2}u(t)$ is bounded in $L^q(\mathbb{S}^2)$ for some $q > 1$. Hence $u(t) \in W^{2,q}(\mathbb{S}^2)$. By Sobolev embedding theorem and the boundedness of area, we have

$$\|u(t)\|_{L^\infty(\mathbb{S}^2)} \le C.$$

Go back to Eq. (2.3), then $u(t)$ is bounded in $H^2(\mathbb{S}^2)$, which contradicts Lemma 6.1. □

We should point out that an alternative proof of Lemma 6.3 for the sequence of solutions is also available in [27, Theorem 3.2].

By Lemma 5.1 and a simple observation that

$$\alpha(t) - 1 = \fint_{\mathbb{S}^2} (\alpha(t) f - K) e^{2u} \mathrm{d}\mu_{\mathbb{S}^2},$$

we easily reach

$$\lim_{t\to\infty} \alpha(t) = 1. \tag{6.1}$$

With this at hand, we can do the estimation

$$\begin{aligned} &\|K - f\|_{L^2(\mathbb{S}^2,g)} \le \|K - \alpha f\|_{L^2(\mathbb{S}^2,g)} + |\alpha - 1| \, \|f\|_{L^2(\mathbb{S}^2,g)} \\ &\le \|K - \alpha f\|_{L^2(\mathbb{S}^2,g)} + C|\alpha - 1| \to 0, \quad \text{as } t \to \infty. \end{aligned}$$

Define

$$S(t)=\frac{\int_{\mathbb{S}^2} x\,\mathrm{d}\mu_{g(t)}}{\int_{\mathbb{S}^2}\mathrm{d}\mu_{g(t)}}$$

to be the center of the metric $g(t)$ and denote its radial image by $Q(t)=S(t)/|S(t)|$ whenever $S(t)\neq 0$.

**Theorem 6.4.** *If $f$ can not be realized as the Gaussian curvature of a conformal metric of $g_{\mathbb{S}^2}$, then, as $t\to\infty$, we have that the flow metric $g(t)$ concentrates at a single point $Q\in\mathbb{S}^2$, and the normalized conformal factor $v(t)\to v_\infty$, as well as the metric $h(t)\to e^{2v_\infty}g_{\mathbb{S}^2}$ in $H^2(\mathbb{S}^2)$, where $f(Q)>0$ and $v_\infty\equiv-[\log f(Q)]/2$. Moreover, $\|\varphi(t)-Q(t)\|_{L^2(\mathbb{S}^2)}\to 0$ as $t\to\infty$, and also $\|f\circ\varphi(t)-f(Q(t))\|_{L^2(\mathbb{S}^2)}\to 0$ and $f(Q(t))\to e^{-2v_\infty}$ as $t\to\infty$.*

**Proof.** Let us refresh ourselves the definition of the conformal transformation $\varphi$: (with $\lambda(t)\geq 1$)

$$u\circ\varphi=v+\log\frac{\lambda^2(1+P(t)\cdot x)+1-P(t)\cdot x}{2\lambda}.$$

Notice that

$$v(P(t))+\log\lambda\leq\max_{\mathbb{S}^2}u\circ\varphi\leq\|v\|_{L^\infty(\mathbb{S}^2)}+\log\lambda$$

and

$$-\|v\|_{L^\infty(\mathbb{S}^2)}-\log\lambda\leq\min_{\mathbb{S}^2}u\circ\varphi\leq v(-P(t))-\log\lambda.$$

We remark that the above two inequalities also can give an alternative proof of Lemma 6.1.

As $\lambda(t)\to+\infty$ as $t\to\infty$, we claim that there is only one concentration point. Assume on the contrary, assume $u(t)$ concentrates at more than one point. Thus we can freely choose two different points $\{P_1,P_2\}$. For $l=1,2$, this implies that

$$\liminf_{t\to\infty}\int_{B_R(P_l)}|K(t)|\mathrm{d}\mu_{g(t)}\geq 2\pi,$$

where $R<\mathrm{dist}_{\mathbb{S}^2}(P_1,P_2)/2$.

Then

$$\liminf_{t\to\infty}\int_{B_R(P_l)}|f|\mathrm{d}\mu_{g(t)}\geq\liminf_{t\to\infty}\int_{B_R(P_l)}|K(t)|\mathrm{d}\mu_{g(t)}-\limsup_{t\to\infty}\int_{\mathbb{S}^2}|K(t)-f|\mathrm{d}\mu_{g(t)}\geq 2\pi. \tag{6.2}$$

Keep in mind that each $P_l$ must be the accumulated point of $\{P(t)\}$. Thus we may choose a subsequence such that $P(t_n)\to P_1$. Then, for any $y\in B_R(P_2)$, we should have $2\geq 1-P(t_n)\cdot y\geq c_1>0$ for all $n\in\mathbb{N}$.

Now by direct computation, for each $y \in B_R(P_2)$,

$$u_{t_n}(y) = v(\varphi^{-1}(y)) + \log \frac{2\lambda}{\lambda^2(1 - P(t_n) \cdot y) + (1 + P(t_n) \cdot y)} \to -\infty.$$

Clearly this implies

$$\lim_{n\to\infty} \int\limits_{B_R(P_2)} |f| e^{2u_{t_n}} \mathrm{d}\mu_{\mathbb{S}^2} = 0$$

which contradicts (6.2) with $l = 2$. Thus our claim follows. Denote the unique concentration point by $Q \in \mathbb{S}^2$. By Lemma 6.2, there exists a function $v_\infty$ such that $v(t) \to v_\infty$ in $C^{1,\gamma/2}(\mathbb{S}^2)$.

Recall that the normalized conformal factor $v$ satisfies

$$-\Delta_{\mathbb{S}^2} v + 1 = K_h e^{2v}.$$

Since $P(t) \to Q$ and $\lambda(t) \to \infty$, we have, in the sense of $L^2(\mathbb{S}^2)$, when $t \to \infty$,

$$K_h = K_h - f_\varphi + f_\varphi - f(Q) + f(Q) \to f(Q),$$

hence $v_\infty$ weakly solves

$$-\Delta_{\mathbb{S}^2} v_\infty + 1 = f(Q) e^{2v_\infty}. \tag{6.3}$$

This implies that $f(Q) > 0$ which can be seen by integrating Eq. (6.3) over $\mathbb{S}^2$ to get $4\pi = f(Q) \int_{\mathbb{S}^2} e^{2v_\infty} \mathrm{d}\mu_{\mathbb{S}^2}$. By the uniqueness theorem in [14] of solutions of (6.3) for solutions which are normalized, we conclude that $v_\infty \equiv -\frac{1}{2} \log f(Q)$. This proves the first claim in the statement. With this ready for use, we have no difficulty to see that as a metric, $h \to e^{2v_\infty} g_{\mathbb{S}^2}$ in $H^2(\mathbb{S}^2)$. Furthermore, let us refresh ourselves first,

$$S(t) = \int\limits_{\mathbb{S}^2} \varphi e^{2\tilde{v}} \mathrm{d}\mu_{\mathbb{S}^2}$$

and $Q(t) = S(t)/|S(t)|$, where $\tilde{v} = v - \frac{1}{2} \log \fint_{\mathbb{S}^2} e^{2v} \mathrm{d}\mu_{\mathbb{S}^2}$. Thus $\tilde{v} \to 0$ in $C^{1,\gamma}(\mathbb{S}^2)$.

Next we observe that, by direct checking, $|S(t) - Q| \to 0$ as $t \to \infty$. Since $|Q| = 1$, hence if $t$ is large enough, then $|S| \neq 0$. Therefore it follows that $1 - |S(t)|^2 = (Q + S(t)) \cdot (Q - S(t)) \to 0$ as $t \to \infty$. Thus,

$$\|\varphi - Q(t)\|_{L^2(\mathbb{S}^2)} \le \|\varphi - S(t)\|_{L^2(\mathbb{S}^2)} + C(1 - |S(t)|) \to 0,$$

as $t \to \infty$.

Finally the boundedness of $C^1$ norm of $f$ clearly implies that $\|f_\varphi - f(Q(t))\|_{L^2(\mathbb{S}^2)} \to 0$ as $t \to \infty$, so is true for $f(Q(t)) - f(Q)$. Now the proof of Theorem 6.4 is completed. □

## 7. Finite-dimensional dynamics

Let $\{\varphi_i^g\}$ be the set of eigen-functions for $-\Delta_g$, that is,

$$-\Delta_g \varphi_i^g = \lambda_i^g \varphi_i^g, \quad i \in \mathbb{N}$$

with eigenvalues $\lambda_0^g = 0 < \lambda_1^g \le \lambda_2^g \le \cdots$. It is well-known that it can be selected such that it is an $L^2(\mathbb{S}^2, V^{-1}\mathrm{d}\mu_g)$ - orthonormal set, where $V$ is the area of the $\mathbb{S}^2$ with respect to the measure $\mathrm{d}\mu_g$. Set $\varphi_i^h = \varphi_i^g \circ \varphi$. Then $\{\varphi_i^h\}$ is an $L^2(\mathbb{S}^2, V^{-1}\mathrm{d}\mu_h)$-orthonormal set. Of course, they are the eigen-functions for $-\Delta_h$ with eigenvalues $\lambda_i^h = \lambda_i^g, \ \forall\, i \in \mathbb{N}$. For comparison, we also list the eigen-functions $\{\varphi_i = \varphi_i^{\mathbb{S}^2}\}$ for $-\Delta_{\mathbb{S}^2}$ with eigenvalues $\lambda_i = \lambda_i^{\mathbb{S}^2}$ which will be labeled as

$$\lambda_0 = 0 < \lambda_1 = \lambda_2 = \lambda_3 = 2 < \lambda_4 = 6 \le \cdots .$$

Surely it will also be an $L^2(\mathbb{S}^2, (4\pi)^{-1}\mathrm{d}\mu_{\mathbb{S}^2})$-orthonormal set.

By Theorem 6.4, we have $\lambda_i^g = \lambda_i^h \to \lambda_i f(Q)$ due to area factor $e^{-2v_\infty} = f(Q)$; moreover, we may arrange the order of the eigenfunctions so that $\varphi_i^h \to \varphi_i$ smoothly as $t \to \infty$.

Expand

$$\alpha f - K = \sum_{i\ge 0} \beta^i \varphi_i^g, \quad \alpha f_\varphi - K_h = \sum_{i\ge 0} \gamma^i \varphi_i^h, \quad \text{and} \quad f = \sum_{i\ge 0} f^i \varphi_i^g$$

with coefficients

$$\beta^i = V^{-1} \int_{\mathbb{S}^2} (\alpha f - K)\varphi_i^g \mathrm{d}\mu_g = V^{-1} \int_{\mathbb{S}^2} (\alpha f_\varphi - K_h)\varphi_i^h \mathrm{d}\mu_h = \gamma^i$$

and

$$f^i = V^{-1} \int_{\mathbb{S}^2} f \varphi_i^g \mathrm{d}\mu_g.$$

In particular, $V^{-1}\int_{\mathbb{S}^2} (\varphi_0^g)^2 \mathrm{d}\mu_g = 1$, together with the fact that $\varphi_0^g$ is a constant, implies that

$$\varphi_0^g = 1.$$

Our normalization $\int_{\mathbb{S}^2} (\alpha f - K) f \mathrm{d}\mu_g = 0$ provides an equation on those coefficients:

$$0 = \sum_{i\ge 0} \beta^i f^i = \beta^0 f^0 + \sum_{i\ge 1} \beta^i f^i .$$

Then, by Cauchy-Schwarz inequality,

$$(f^0\beta^0)^2 = (\sum_{i\ge 1} \beta^i f^i)^2$$

$$\leq (\sum_{i\geq 1}(\beta^i)^2)(\sum_{i\geq 1}(f^i)^2)$$
$$\leq V^{-1}\int_{\mathbb{S}^2}(\alpha f-K)^2 d\mu_g)(V^{-1}\int_{\mathbb{S}^2} f^2 d\mu_g-(f^0)^2).$$

By Theorem 6.4 and $f^0=\int_{\mathbb{S}^2} f\varphi_0^g V^{-1}d\mu_g=4\pi V^{-1}$, we have, as $t\to\infty$,

$$\int_{\mathbb{S}^2} f^2V^{-1}d\mu_g-(f^0)^2=V^{-1}[\int_{\mathbb{S}^2} f^2 d\mu_g-\frac{(4\pi)^2}{V}]\to f(Q)[f(Q)-f(Q)]=0,$$

since $V\to\frac{4\pi}{f(Q)}$ as $t\to\infty$. This, together with (2.6) and the fact that $f(Q)>0$, gives

$$(\beta^0)^2=o(1)V^{-1}\int_{\mathbb{S}^2}(\alpha f-K)^2 d\mu_g. \tag{7.1}$$

Recall that

$$F_2(t)=\int_{\mathbb{S}^2}(\alpha f-K)^2 d\mu_g=\int_{\mathbb{S}^2}(\alpha f_\varphi-K_h)^2 d\mu_h,$$
$$G_2(t)=\int_{\mathbb{S}^2}|\nabla(\alpha f-K)|_g^2 d\mu_g=\int_{\mathbb{S}^2}|\nabla(\alpha f_\varphi-K_h)|_h^2 d\mu_h.$$

**Lemma 7.1.** *With error $o(1)\to 0$ as $t\to\infty$, there holds*

$$\frac{d}{dt}F_2(t)=-(2+o(1))G_2(t)+(4f(Q)+o(1))F_2(t).$$

**Proof.** By the formula (5.3), we have

$$\frac{dF_2}{dt}=-2G_2+4\alpha\int_{\mathbb{S}^2} f_\varphi(\alpha f_\varphi-K_h)^2 d\mu_h+2\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^3 d\mu_h$$
$$=-2G_2+4\alpha f(Q(t))F_2+4\alpha\int_{\mathbb{S}^2}(f_\varphi-f(Q(t)))(\alpha f_\varphi-K_h)^2 d\mu_h+2\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^3 d\mu_h.$$

By Theorem 6.4 we have, as $t\to\infty$,

$$\left|\int_{\mathbb{S}^2}(f_\varphi-f(Q(t)))(\alpha f_\varphi-K_h)^2 d\mu_h\right|\leq C\left\|f_\varphi-f(Q(t))\right\|_{L^2(\mathbb{S}^2,h)}\left\|\alpha f_\varphi-K_h\right\|^2_{L^4(\mathbb{S}^2,h)}$$
$$=o(1)(F_2+G_2),$$

and

$$\left|\int_{\mathbb{S}^2}(K_h-\alpha f_\varphi)^3 \mathrm{d}\mu_h\right| \le C \left\|K_h-\alpha f_\varphi\right\|_{L^2(\mathbb{S}^2,h)} \|K-\alpha f\|^2_{L^4(\mathbb{S}^2,h)} = o(1)(F_2+G_2),$$

where we have used $F_2(t)=o(1)$ and $|f(Q(t))-f(Q)|=o(1)$. We just reminder the reader that the estimate on the $L^4(\mathbb{S}^2)$ norm of $K-\alpha f$ has been treated in our Lemma 5.3. We should not repeat it here again. Thus this lemma follows. □

By the estimate (7.1), we have

$$G_2 = V\sum_{i\ge 1}\lambda_i^g|\beta^i|^2 \ge \lambda_1^g\sum_{i\ge 1} V|\beta^i|^2 = (\lambda_1^g+o(1))F_2 = (2f(Q)+o(1))F_2.$$

Let $x$ be the position vector of $\mathbb{S}^2$ in $\mathbb{R}^3$. Let $\{e_1,e_2,e_3\}$ be an orthonormal basis of $\mathbb{R}^3$. Then we may choose $\varphi_i=\sqrt{3}x\cdot e_i$ as $L^2(\mathbb{S}^2,(4\pi)^{-1}g_{\mathbb{S}^2})$-orthonormal basis for the corresponding eigen-space of $-\Delta_{\mathbb{S}^2}$ with eigenvalue 2. Notice that the $e_i$ for $i=1,2,3$ is chosen so that $\varphi_i^h\to\varphi_i$ as $t\to\infty$.

Define

$$b = V^{-1}\int_{\mathbb{S}^2} x(\alpha f_\varphi - K_h)\mathrm{d}\mu_h$$

and write $b=(b\cdot e_1, b\cdot e_2, b\cdot e_3)$.

**Lemma 7.2.** *With error $o(1)\to 0$ as $t\to\infty$,*

$$\frac{\mathrm{d}b}{\mathrm{d}t} = o(1)[F_2(t)^{\frac{1}{2}}+G_2(t)^{\frac{1}{2}}].$$

**Proof.** By Eq. (2.1), we have

$$\alpha\int_{\mathbb{S}^2} f^2\mathrm{d}\mu_g = \int_{\mathbb{S}^2} fK\mathrm{d}\mu_{\mathbb{S}^2} = \int_{\mathbb{S}^2} f(-\Delta_{\mathbb{S}^2}u+1)\mathrm{d}\mu_{\mathbb{S}^2} = -\int_{\mathbb{S}^2} u\Delta_{\mathbb{S}^2} f\mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} f\mathrm{d}\mu_{\mathbb{S}^2}.$$

Differentiate both sides of the above equation with respect to $t$ to show

$$\alpha_t\int_{\mathbb{S}^2} f^2\mathrm{d}\mu_g = \int_{\mathbb{S}^2}\langle\nabla f,\nabla(\alpha f-K)\rangle_g\mathrm{d}\mu_g + 2\alpha\int_{\mathbb{S}^2} f^2(K-\alpha f)\mathrm{d}\mu_g.$$

Notice that, with the observation that in dimension two, Dirichlet energy is conformally invariant,

$$\left|\int_{\mathbb{S}^2}\langle\nabla f,\nabla(\alpha f-K)\rangle_g\mathrm{d}\mu_g\right| \le G_2^{\frac{1}{2}}\|\nabla f\|_{L^2(\mathbb{S}^2)} \le CG_2^{\frac{1}{2}}.$$

These, together with (2.7), give

$$|\alpha_t| \le C(G_2^{1/2} + F_2^{1/2}).$$

By Theorem 6.4 and (6.1) we have

$$\begin{aligned}
&\lim_{t\to\infty} \int_{\mathbb{S}^2} (\alpha f_\varphi - e^{-2v})^2 \mathrm{d}\mu_h \\
=&\lim_{t\to\infty} [\alpha^2 \int_{\mathbb{S}^2} f_\varphi^2 \mathrm{d}\mu_h - 2\alpha \int_{\mathbb{S}^2} f_\varphi \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} e^{-2v}\mathrm{d}\mu_{\mathbb{S}^2}] \\
=&4\pi[f(Q) - 2f(Q) + e^{-2v_\infty}] = 0.
\end{aligned} \tag{7.2}$$

Recall Eq. (4.8) and the estimate (4.9), we have

$$\left|(\frac{\mathrm{d}\varphi^{-1}}{\mathrm{d}t})\circ\varphi\right| \le \left|\frac{\lambda_t}{\lambda}\right| + \frac{\lambda^2-1}{\lambda}\left|\frac{\mathrm{d}P}{\mathrm{d}t}\right| \le CF_2(t)^{\frac{1}{2}},$$

where we have observed that $(\lambda-1)^2 = \lambda^2 - 1 - 2(\lambda - 1) \le \lambda^2 - 1$ due to the fact that $\lambda \ge 1$. This observation has been used in the estimate of the second term in the estimate (4.9).

With the above inequality, Eq. (5.1) and the fact that $\fint_{\mathbb{S}^2} \varphi^{-1} e^{2u}\mathrm{d}\mu_{\mathbb{S}^2} = 0$, it is straightforward to show

$$\begin{aligned}
&V\frac{\mathrm{d}b}{\mathrm{d}t} + \frac{\mathrm{d}V}{\mathrm{d}t}b \\
=&\int_{\mathbb{S}^2} \frac{\partial\varphi^{-1}}{\partial t}(\alpha f - K)e^{2u}\mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} \varphi^{-1}(\alpha_t f - K_t)e^{2u}\mathrm{d}\mu_{\mathbb{S}^2} + 2\int_{\mathbb{S}^2} \varphi^{-1}(\alpha f - K)^2 e^{2u}\mathrm{d}\mu_{\mathbb{S}^2} \\
=&\int_{\mathbb{S}^2} (\frac{\partial\varphi^{-1}}{\partial t})\circ\varphi(\alpha f_\varphi - K_h)\mathrm{d}\mu_h + \alpha_t \int_{\mathbb{S}^2} \varphi^{-1}(f - f(Q(t))\mathrm{d}\mu_g \\
&+ 2\int_{\mathbb{S}^2} \varphi^{-1}\alpha f(\alpha f - K)\mathrm{d}\mu_g + \int_{\mathbb{S}^2} \varphi^{-1}\Delta_{\mathbb{S}^2}(K - \alpha f)\mathrm{d}\mu_{\mathbb{S}^2} \\
=&o(1)(F_2(t)^{\frac{1}{2}} + G_2(t)^{\frac{1}{2}}) + 2\int_{\mathbb{S}^2} x\alpha f_\varphi(\alpha f_\varphi - K_h)e^{2v}\mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} x\Delta_{\mathbb{S}^2}(K_h - \alpha f_\varphi)\mathrm{d}\mu_{\mathbb{S}^2} \\
=&o(1)(F_2(t)^{\frac{1}{2}} + G_2(t)^{\frac{1}{2}}) + 2\int_{\mathbb{S}^2} x(\alpha f_\varphi - e^{-2v})(\alpha f_\varphi - K_h)e^{2v}\mathrm{d}\mu_{\mathbb{S}^2} \\
=&o(1)(F_2(t)^{\frac{1}{2}} + G_2(t)^{\frac{1}{2}}),
\end{aligned}$$

where the last identity follows from (7.2). This together with a simple observation that $|\frac{\mathrm{d}V}{\mathrm{d}t}b| = O(1)F_2$ yields the desired estimate. $\square$

We try to avoid the trouble cost by the normalization and set

$$B = \sqrt{3}b.$$

It follows that, for $1 \leq i \leq 3$, there holds

$$\begin{aligned}|\beta^i - B^i|^2 =& \left| V^{-1} \int_{\mathbb{S}^2} (\alpha f_\varphi - K_h)\varphi_i^h \mathrm{d}\mu_h - V^{-1} \int_{\mathbb{S}^2} \varphi_i (\alpha f_\varphi - K_h) \mathrm{d}\mu_h \right|^2 \\ \leq& [V^{-1} \int_{\mathbb{S}^2} (\varphi_i^h - \varphi_i)^2 \mathrm{d}\mu_h] \cdot [V^{-1} \int_{\mathbb{S}^2} (\alpha f_\varphi - K_h)^2 \mathrm{d}\mu_h] \\ =& o(1) V^{-1} F_2. \end{aligned} \tag{7.3}$$

Our next task is to estimate $F_2$ in terms of $B$, namely, we have the following lemma.

**Lemma 7.3.** *With error $o(1) \to 0$ as $t \to \infty$, we have $F_2 = (1 + o(1))V|B|^2$.*

**Proof.** By the estimates (7.1) and (7.3), we have

$$F_2 = V \sum_{i\geq 0} |\beta^i|^2 = V|B|^2 + V \sum_{i\geq 4} |\beta^i|^2 + o(1)F_2.$$

Let $\hat{F}_2 = V \sum_{i\geq 4} |\beta^i|^2$. Let us first assume that $V|B(t_1)|^2 \geq \hat{F}_2(t_1)$ for some large $t_1 \geq 0$. For $t$ near $t_1$, we may expand $F_2$ as

$$F_2 = (1 + \delta(t))V|B|^2 \implies \hat{F}_2 = \delta(t)V|B|^2 + o(1)F_2 = (\delta(t) + o(1))V|B|^2,$$

by assuming that $-1/2 < \delta(t) \leq 2$ if $t_1$ is sufficiently large. Then

$$\frac{\mathrm{d}}{\mathrm{d}t} F_2 = \frac{\mathrm{d}\delta}{\mathrm{d}t} V|B|^2 + 2(1+\delta)VB \cdot \frac{\mathrm{d}B}{\mathrm{d}t} + (1 + \delta(t))\frac{\mathrm{d}V}{\mathrm{d}t}|B|^2. \tag{7.4}$$

By Lemma 7.2 and the observation that $V^{-1}\frac{\mathrm{d}V}{\mathrm{d}t} = o(1)$, we estimate

$$|(1+\delta(t))\frac{\mathrm{d}V}{\mathrm{d}t}|B|^2| + 2(1+\delta(t))V \left| B \cdot \frac{\mathrm{d}B}{\mathrm{d}t} \right| \leq o(1)(G_2 + F_2).$$

Inserting the above inequality into (7.4), by Lemma 7.1 we obtain

$$\begin{aligned} V\frac{\mathrm{d}\delta}{\mathrm{d}t}|B|^2 =& -(2 + o(1))G_2 + (4f(Q) + o(1))F_2 \\ \leq& -2V \sum_{i\geq 0} (f(Q)\lambda_i - 2f(Q) + o(1))|\beta_i|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq -8f(Q)V\sum_{i\geq 4}|\beta^i|^2 + o(1)F_2 \\
&= -8f(Q)\hat{F}_2 + o(1)F_2 \\
&= -(8f(Q)\delta + o(1))V|B|^2.
\end{aligned}
$$

Dividing both sides by the (non-vanishing) term $V|B|^2$ to show

$$\frac{\mathrm{d}\delta}{\mathrm{d}t} \leq -(8f(Q)\delta + o(1)).$$

It follows from this that

$$\delta(t) \leq \delta(t_1)e^{-8f(Q)t} + e^{-8f(Q)t}\int_{t_1}^{t} o(1)e^{8f(Q)\tau}\mathrm{d}\tau,$$

which implies that $\delta(t) \to 0$ as $t \to \infty$ by applying the L'Hospital's rule on the second term on the right hand side. This in turn implies that

$$F_2 = (1 + o(1))V|B|^2$$

as required.

It remains to show that for any given $T > 0$, there always exists a time $t_1 > T$ such that $V(t_1)|B(t_1)|^2 \geq \hat{F}_2(t_1)$.

However, assuming that $V|B|^2 \leq \hat{F}_2$ for all sufficiently large $t$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}F_2 \leq -8f(Q)\hat{F}_2 + o(1)F_2 \leq (-8f(Q) + o(1))F_2.$$

Therefore, we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}F_2 \leq -4f(Q)F_2.$$

Then, it follows that

$$F_2(t) \leq Ce^{-4f(Q)t}$$

for some constant $C > 0$ and all $t \geq 0$.

But then for any $r_0 > 0$ and any $x \in \mathbb{S}^2$, we can estimate

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Vol}(B_{r_0}(x), g)\right| = 2\left|\int_{\mathbb{S}^2} u_t \mathrm{d}\mu_g\right| \leq C\|u_t\|_{L^2(\mathbb{S}^2,g)} \leq C(V)e^{-2f(Q)t},$$

where $B_{r_0}(x) = B_{r_0}(x, g_{\mathbb{S}^2})$, and we find that, since $f(Q) > 0$ and is independent of $t$,

$$\mathrm{Vol}(B_{r_0}(x), g) \leq \mathrm{Vol}(B_{r_0}(x), g(t_1)) + Ce^{-C_1 t} < \frac{\pi}{f(Q)}$$

uniformly for $t \geq t_1$ if we first choose sufficiently large $t_1 \geq 0$ and the $r_0 > 0$ sufficiently small. However, from Theorem 6.4 we infer that for the accumulation point $Q$ of $Q(t)$ and any $r_0 > 0$,

$$\mathrm{Vol}(B_{r_0}(Q), g) \to \frac{4\pi}{f(Q)}$$

as $t \to \infty$. This contradiction proves the claim, and hence one completes the proof of Lemma. $\square$

Recall that

$$S(t) = \frac{\int_{\mathbb{S}^2} x \mathrm{d}\mu_g}{\int_{\mathbb{S}^2} \mathrm{d}\mu_g} = \frac{\int_{\mathbb{S}^2} \varphi \mathrm{d}\mu_h}{\int_{\mathbb{S}^2} \mathrm{d}\mu_h}.$$

Then it follows from Theorem 6.4 that $h(t) \to e^{2v_\infty} g_{\mathbb{S}^2}$ in $H^2(\mathbb{S}^2)$ and $|S(t)| \to 1$ in $L^2(\mathbb{S}^2)$ as $t \to \infty$. This implies that the shadow flow of $S$, defined by

$$\Theta(t) := \fint_{\mathbb{S}^2} \varphi(t) \mathrm{d}\mu_{\mathbb{S}^2}$$

approximates to $Q(t)$ in $L^2(\mathbb{S}^2)$ sense.

For convenience in later argument, we extend $f(sx) = f(x)$ for $0 < s < 1, x \in \mathbb{S}^2$.

**Lemma 7.4.** *With a uniform constant $C > 0$, there holds*

$$\|v - \tilde{v}_\infty(t)\|_{H^2(\mathbb{S}^2)} \leq C(F_2^{\frac{1}{2}} + \left\| f_\varphi - f(\Theta) \right\|_{L^2(\mathbb{S}^2)}),$$

*where* $\tilde{v}_\infty(t) = \log \fint_{\mathbb{S}^2} e^{2u} \mathrm{d}\mu_{\mathbb{S}^2}/2$.

**Proof.** First notice that as $t \to \infty$,

$$\|v - \tilde{v}_\infty(t)\|_{H^2(\mathbb{S}^2)} \leq \|v - v_\infty\|_{H^2(\mathbb{S}^2)} + \|v_\infty - \tilde{v}_\infty(t)\|_{H^2(\mathbb{S}^2)} \to 0$$

by virtue of Theorem 6.4. Expanding

$$e^{2(v-\tilde{v}_\infty(t))} - 1 = \sum_{i=0}^{\infty} V^i \varphi_i \quad \text{and} \quad v - \tilde{v}_\infty(t) = \sum_{i=0}^{\infty} v^i \varphi_i.$$

By (2.4) we have

$$V^0 = \fint_{\mathbb{S}^2} (e^{2(v-\tilde{v}_\infty(t))} - 1) \mathrm{d}\mu_{\mathbb{S}^2} = 0,$$

as well as $V^i = 0$, $1 \le i \le 3$ by the normalization (4.1). Moreover, by Taylor expansion,

$$\begin{aligned}
V^i =& \fint_{\mathbb{S}^2} (e^{2(v-\tilde{v}_\infty(t)} - 1)\varphi_i \mathrm{d}\mu_{\mathbb{S}^2} \\
=& 2\fint_{\mathbb{S}^2} (v - \tilde{v}_\infty(t))\varphi_i \mathrm{d}\mu_{\mathbb{S}^2} + O(\|v - \tilde{v}_\infty(t)\|^2_{L^\infty(\mathbb{S}^2)}) \\
=& 2v^i + o(1)\|v - \tilde{v}_\infty(t)\|_{H^2(\mathbb{S}^2)}.
\end{aligned}$$

Thus $V^i = 0$ implies that

$$|v^i| = o(1)\,\|v - \tilde{v}_\infty(t)\|_{H^2(\mathbb{S}^2)}\,, \quad 0 \le i \le 3. \tag{7.5}$$

Rewrite the Gaussian curvature equation as

$$-\Delta_{\mathbb{S}^2} v = K_h e^{2v} - 1 = (K_h - \alpha f_\varphi)e^{2v} + \alpha(f_\varphi - f(\Theta))e^{2v} + (\alpha - 1)f(\Theta)e^{2v} + f(\Theta)e^{2v} - 1.$$

This, together with the fact that $|\alpha - 1| \le C F_2^{1/2}$, implies that

$$\|\Delta_{\mathbb{S}^2} v\|^2_{L^2(\mathbb{S}^2)} \le C(\|K_h - \alpha f_\varphi\|^2_{L^2(\mathbb{S}^2,h)} + \|f_\varphi - f(\Theta)\|^2_{L^2(\mathbb{S}^2)}) + 2\int_{\mathbb{S}^2} (f(\Theta)e^{2v} - 1)^2 \mathrm{d}\mu_{\mathbb{S}^2}.$$

On the one hand, we notice that

$$\begin{aligned}
& f(\Theta)e^{2v} - 1 \\
=& f(\Theta)e^{2\tilde{v}_\infty(t)}(e^{2(v-\tilde{v}_\infty(t))} - 1) + f(\Theta)e^{2\tilde{v}_\infty(t)} - \fint_{\mathbb{S}^2} f_\varphi e^{2v} \mathrm{d}\mu_{\mathbb{S}^2} \\
=& f(\Theta)e^{2\tilde{v}_\infty(t)}(e^{2(v-\tilde{v}_\infty(t))} - 1) + \fint_{\mathbb{S}^2} f(\Theta)(e^{2\tilde{v}_\infty(t)} - e^{2v})\mathrm{d}\mu_{\mathbb{S}^2} + \fint_{\mathbb{S}^2} (f(\Theta) - f_\varphi)e^{2v} \mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned}$$

It follows from this, with the help of Hölder's inequality, the following estimate holds true

$$\begin{aligned}
& \fint_{\mathbb{S}^2} (f(\Theta)e^{2v} - 1)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
\le& 3f(\Theta)^2 e^{4\tilde{v}_\infty(t)} \fint_{\mathbb{S}^2} (e^{2(v-\tilde{v}_\infty(t))} - 1)^2 \mathrm{d}\mu_{\mathbb{S}^2} + 3f^2(\Theta)e^{4\tilde{v}_\infty(t)} \left( \fint_{\mathbb{S}^2} (1 - e^{2(v-\tilde{v}_\infty(t))})\mathrm{d}\mu_{\mathbb{S}^2} \right)^2 \\
& + 3\left( \fint_{\mathbb{S}^2} (f(\Theta) - f_\varphi)e^{2v} \mathrm{d}\mu_{\mathbb{S}^2} \right)^2
\end{aligned}$$

$$\leq 6 f(\Theta)^2 e^{4\tilde{v}_\infty(t)} \fint_{\mathbb{S}^2} (e^{2(v-\tilde{v}_\infty(t))} - 1)^2 \mathrm{d}\mu_{\mathbb{S}^2} + C \left\| f(\Theta) - f_\varphi \right\|^2_{L^2(\mathbb{S}^2)} .$$

Next the inequality

$$\left\| e^{2(v-\tilde{v}_\infty(t))} - 1 \right\|^2_{L^2(\mathbb{S}^2)} \leq (4 + o(1)) \left\| v - \tilde{v}_\infty(t) \right\|^2_{L^2(\mathbb{S}^2)}$$

can be used to bound

$$\left\| \Delta_{\mathbb{S}^2} v \right\|^2_{L^2(\mathbb{S}^2)} \leq C(\left\| \alpha f_\varphi - K_h \right\|^2_{L^2(\mathbb{S}^2,h)} + \left\| f_\varphi - f(\Theta) \right\|^2_{L^2(\mathbb{S}^2)}) + (24 + o(1)) \left\| v - \tilde{v}_\infty(t) \right\|^2_{L^2(\mathbb{S}^2)} . \tag{7.6}$$

On the other hand, by the estimate (7.5), noticing that the zero eigenvalue $\lambda_0 = 0$, we have

$$\left\| \Delta_{\mathbb{S}^2} v \right\|^2_{L^2(\mathbb{S}^2)} = \sum_{i=1}^{\infty} \lambda_i^2 |v^i|^2 \geq \lambda_4^2 \| v - \tilde{v}_\infty(t) \|^2_{L^2(\mathbb{S}^2)} + o(1) \left\| v - \tilde{v}_\infty(t) \right\|^2_{H^2(\mathbb{S}^2)} .$$

Since $24 < \lambda_4^2 = 36$, for sufficiently large $t > 0$ we can absorb the last term on the right hand side of Eq. (7.6) into the left hand side, and thus obtain

$$\begin{aligned} \| v - \tilde{v}_\infty(t) \|_{H^2(\mathbb{S}^2)} \leq & C(\left\| \Delta_{\mathbb{S}^2} v \right\|_{L^2(\mathbb{S}^2)} + \| v - \tilde{v}_\infty(t) \|_{L^2(\mathbb{S}^2)}) \\ \leq & C(\left\| \alpha f_\varphi - K_h \right\|_{L^2(\mathbb{S}^2)} + \left\| f_\varphi - f(\Theta) \right\|_{L^2(\mathbb{S}^2)}) \end{aligned}$$

as precisely we claimed. □

Now we expand the vector $b$ in terms of the order of $\lambda$. In fact, we have the following lemma.

**Lemma 7.5.** *With error $o(1) \to 0$ as $t \to \infty$, for some uniform constants $A_1, A_2 > 0$, there hold*

$$\begin{aligned} b - (b \cdot P)P =& A_1 \frac{\alpha}{V} \frac{1}{\lambda} \nabla f(P) + O(\frac{\log \lambda}{\lambda^2}), \\ b \cdot P =& - A_2 \frac{\alpha}{V} \frac{\log \lambda}{\lambda^2} (\Delta_{\mathbb{S}^2} f(P) + O(|\nabla f(P)|^2_{\mathbb{S}^2}) + o(1)). \end{aligned}$$

**Proof.** Using Eq. $-\Delta_{\mathbb{S}^2} x = 2x$ and an integration by parts, we rewrite

$$\begin{aligned} 2b = 2V^{-1} \int_{\mathbb{S}^2} x(\alpha f_\varphi - K_h) \mathrm{d}\mu_h =& -V^{-1} \int_{\mathbb{S}^2} \Delta_{\mathbb{S}^2} x (\alpha f_\varphi - K_h) \mathrm{d}\mu_h \\ =& V^{-1} \int_{\mathbb{S}^2} \langle \nabla x, \nabla(\alpha f_\varphi - K_h) \rangle_{\mathbb{S}^2} \mathrm{d}\mu_h + R_1 \end{aligned}$$

with error

$$R_1 = 2V^{-1}\int\limits_{\mathbb{S}^2}\langle\nabla x, \nabla(v-\tilde{v}_\infty(t))\rangle_{\mathbb{S}^2}(\alpha f_\varphi - K_h)e^{2v}\mathrm{d}\mu_{\mathbb{S}^2}$$

bounded by

$$|R_1| \le C\,\|v-\tilde{v}_\infty(t)\|_{H^1(\mathbb{S}^2)}\,\left\|\alpha f_\varphi - K_h\right\|_{L^2(\mathbb{S}^2,h)}.$$

Apply the Kazdan-Warner condition to obtain

$$2b = \alpha V^{-1}\int\limits_{\mathbb{S}^2}\langle\nabla x, \nabla f_\varphi\rangle_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} + R_1 + R_2,$$

where

$$\begin{aligned}R_2 =&\alpha V^{-1}\int\limits_{\mathbb{S}^2}\left\langle\nabla x, \nabla f_\varphi\right\rangle_{\mathbb{S}^2}(e^{2v}-e^{2\tilde{v}_\infty(t)})\mathrm{d}\mu_{\mathbb{S}^2}\\ =&2\alpha V^{-1}\int\limits_{\mathbb{S}^2}x(f_\varphi - f(\Theta))(e^{2v}-e^{2\tilde{v}_\infty(t)})\mathrm{d}\mu_{\mathbb{S}^2}\\ &-2\alpha V^{-1}\int\limits_{\mathbb{S}^2}\langle\nabla x, \nabla v\rangle_{\mathbb{S}^2}(f_\varphi - f(\Theta))\mathrm{d}\mu_h,\end{aligned}$$

which can be controlled as follows

$$\begin{aligned}|R_2| &\le C(\|v-\tilde{v}_\infty(t)\|_{L^\infty(\mathbb{S}^2)} + \|v-\tilde{v}_\infty(t)\|_{H^1(\mathbb{S}^2)})\left\|f_\varphi - f(\Theta)\right\|_{L^2(\mathbb{S}^2)}\\ &\le C\,\|v-\tilde{v}_\infty(t)\|_{H^2(\mathbb{S}^2)}\left\|f_\varphi - f(\Theta)\right\|_{L^2(\mathbb{S}^2)}.\end{aligned}$$

Thus, with $R(b) = (R_1+R_2)/2$, we have

$$b = \alpha V^{-1}\int\limits_{\mathbb{S}^2}x(f_\varphi(x) - f(P))\mathrm{d}\mu_{\mathbb{S}^2} + R(b). \tag{7.7}$$

By Lemmas 7.3 and 7.4, we obtain

$$|R(b)| \le C(F_2 + \left\|f_\varphi - f(\Theta)\right\|^2_{L^2(\mathbb{S}^2)}).$$

For computation, we have to choose the local parametrization of $\mathbb{S}^2$. To do so, we take $e_3 = P$ and $e_1, e_2$ are two orthogonal unit vectors in $\mathbb{R}^3$ which are both perpendicular to the $e_3$ and write $x = x^1e_1 + x^2e_2 + x^3e_3$. Clearly $x^3 = x\cdot P$. Let $(y^1, y^2)$ be the local coordinates of $\mathbb{R}^2$. Through stereographic projection $\pi : \mathbb{S}^2 \to \mathbb{R}^2$, we have the natural parametrization of $\mathbb{S}^2$:

$$x^1 = \frac{2y^1}{1+|y|^2},\quad x^2 = \frac{2y^2}{1+|y|^2},\quad x^3 = \frac{1-|y|^2}{1+|y|^2}.$$

Let $D := \{y \in \mathbb{R}^2; |y| \le C_1\}$ for some small constant $C_1 > 0$. We take the Taylor expansion of $f$ around $P$ in this coordinate system

$$f(y) = f(y^1, y^2) = f(P) + \sum_{i=1}^{2} \frac{\partial f}{\partial y^i}(P) y^i + \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial y^i \partial y^j}(P) y^i y^j + o(\sum_{i=1}^{2} |y^i|^2).$$

Denote the plane set $\{y \in \mathbb{R}^2; |y| \le \lambda C_1\}$ by $D_\lambda$. Then it follows that $\pi \circ \phi_{P,\lambda} \circ \pi^{-1}(D_\lambda) = D$. Therefore, observing that, in this coordinate system, the area element is given by $\mathrm{d}A(y) = \frac{4}{(1+|y|^2)^2}\mathrm{d}y$, we have

$$\int\limits_{D_\lambda^c} \frac{4}{(1+|y|^2)^2} \mathrm{d}y = 8\pi \int\limits_{\lambda C_1}^{\infty} \frac{r}{(1+r^2)^2} \mathrm{d}r = \frac{4\pi C_1^2}{1+\lambda^2 C_1^2} = O(\frac{1}{\lambda^2})$$

as $t \to \infty$.

Notice that, again in terms of the coordinates $(y^1, y^2)$, $\varphi_{P,\lambda}$ can be written as

$$\varphi_{P,\lambda}(\pi^{-1}(y)) = \left( \frac{2\lambda y^1}{\lambda^2 + |y|^2}, \frac{2\lambda y^2}{\lambda^2 + |y|^2}, \frac{\lambda^2 - |y|^2}{\lambda^2 + |y|^2} \right).$$

By $\pi(x^1, x^2, x^3) = (\frac{x^1}{1+x^3}, \frac{x^2}{1+x^3})$, one also has

$$f \circ \varphi_{P,\lambda}(\pi^{-1}(y)) = f(\frac{y^1}{\lambda}, \frac{y^2}{\lambda}).$$

A direct computation yields that, by definition of $\Theta$,

$$\Theta(t) = \left(1 - \frac{4\lambda^2}{(1-\lambda^2)^2} \log \lambda + \frac{2}{\lambda^2 - 1}\right) P = (1 - \frac{4\log\lambda}{\lambda^2} + O(\frac{1}{\lambda^2}))P. \tag{7.8}$$

Thus, by our convention, we have $f(\Theta) = f(P)$.

Now in local coordinates, we can write the key part in the expression $b$ as

$$\begin{aligned}
&\int\limits_{\mathbb{S}^2} (f(\varphi_{P,\lambda}(x)) - f(P)) x \mathrm{d}\mu_{\mathbb{S}^2} \\
&= \int\limits_{D_\lambda} (f(\varphi_{P,\lambda}(\pi^{-1}(y)) - f(P)) x(y) \mathrm{d}A(y) + O(\frac{1}{\lambda^2}) \\
&= \int\limits_{D_\lambda} \left[ \sum_{j=1}^{2} \frac{\partial f}{\partial y^j}(P) \frac{y^j}{\lambda} + \frac{1}{2} \sum_{i,j=1}^{2} \frac{\partial^2 f}{\partial y^i \partial y^j}(P) \frac{y^i}{\lambda}\frac{y^j}{\lambda} \right] x(y) \mathrm{d}A(y) + E_\lambda + O(\frac{1}{\lambda^2}),
\end{aligned}$$

where, by smoothness of the function $f$,

$$|E_\lambda| = O(\int\limits_{D_\lambda} \left|\frac{y}{\lambda}\right|^3 \mathrm{d}A(y)) = O(\frac{1}{\lambda^2}).$$

By rotational symmetry, if $1 \le i \ne j \le 3$,

$$\int\limits_{D_\lambda} y^i x^j(y)\mathrm{d}A(y) = 0.$$

It follows from this, for $i = 1, 2$

$$\frac{1}{\lambda}\int\limits_{D_\lambda} \frac{2(y^i)^2}{1+|y|^2}\mathrm{d}A(y) = \frac{2\pi}{\lambda} + O(\frac{1}{\lambda^2}).$$

Moreover, by rotational symmetry and direct computation we have

$$\frac{1}{\lambda^2}\int\limits_{D_\lambda} y^j y^l x^i(y)\mathrm{d}A(y) = \begin{cases} 0, & 1 \le i, j, l \le 2, \\ 0, & i = 3,\ j \ne l, \\ -\frac{4\pi\log\lambda}{\lambda^2} + O(\frac{1}{\lambda^2}), & i = 3,\ 1 \le j = l \le 2, \end{cases}$$

where we only need to check

$$\begin{aligned} &\int\limits_{D_\lambda} x^3(y)(y^j)^2\mathrm{d}A(y) \\ =&\int\limits_{D_\lambda} \frac{(1-|y|^2)(y^j)^2}{(1+|y|^2)}\mathrm{d}A(y) \\ =&2\pi\int\limits_0^{C_1^2\lambda^2} \frac{r(1-r)}{(1+r)^3}\mathrm{d}r \\ =&-4\pi\log\lambda + O(1). \end{aligned}$$

Therefore, we have obtained

$$b = \alpha V^{-1}[\frac{2\pi}{\lambda}\nabla f(P) - \frac{\pi}{2}(\Delta_{\mathbb{S}^2} f)(P)\frac{\log\lambda}{\lambda^2}P] + CR(b) + O(\frac{1}{\lambda^2}). \tag{7.9}$$

With $A_1 = 2\pi$ and $A_2 = \pi/2$, the leading terms in our claim hold true. Now we have to deal with the error terms. To see this, observe that

$$\begin{aligned} \|f_\varphi - f(\Theta)\|^2_{L^2(\mathbb{S}^2)} &= \frac{C}{\lambda^2}|\nabla f|^2_{\mathbb{S}^2}(P)\int\limits_{\mathbb{R}^2}\sum_{i=1}^2 (y^i)^2\mathrm{d}A(y) + C|\nabla f|^2_{\mathbb{S}^2}(P)|\Theta - P|^2 + O(\frac{1}{\lambda^2}) \\ &= C|\nabla f|^2_{\mathbb{S}^2}(P)\frac{\log\lambda}{\lambda^2} + O(\frac{1}{\lambda^2}) \end{aligned} \tag{7.10}$$

where we have used the estimate (7.8), and notice that $F_2 = 3(1+o(1))V|b|^2$, thus, by Eq. (7.7),

$$\begin{aligned}|b|^2 =&|b-(b\cdot P)P|^2+(b\cdot P)^2\\ =&C|R(b)|^2+C|\nabla f|^2_{\mathbb{S}^2}(P)\frac{1}{\lambda^2}+\left(\frac{\pi\alpha}{2V}\right)^2(\Delta_{\mathbb{S}^2}f(P))^2(\frac{(\log\lambda)^2}{\lambda^4}+O(\frac{1}{\lambda^4}))\\ =&C|b|^4+\left(\frac{\pi\alpha}{2V}\right)^2[(\Delta_{\mathbb{S}^2}f(P))^2+O(1)|\nabla f|^2_{\mathbb{S}^2}(P)+o(1)](\frac{(\log\lambda)^2}{\lambda^4}+O(\frac{1}{\lambda^4})).\end{aligned}$$

Now since $|b|^2\to 0$ as $t\to\infty$, if $t$ is large enough, then we have $C|b|^2\le\frac{1}{2}$. Thus we conclude that

$$|b|^2=\frac{1}{2}\left(\frac{\pi\alpha}{V}\right)^2[(\Delta_{\mathbb{S}^2}f(P))^2+O(1)(|\nabla f|^2_{\mathbb{S}^2}(P))^2](\frac{(\log\lambda)^2}{\lambda^4}+O(\frac{1}{\lambda^4})).$$

This implies that $|R(b)|=O(\lambda^{-2}\log\lambda)$. Thus our claim holds. This completes the proof. □

The more precise estimate on the shadow flow is needed.

**Lemma 7.6.** *With $o(1)\to 0$ as $t\to\infty$, we have*

$$(1-|\Theta|^2)=2(4+o(1))\frac{\log\lambda}{\lambda^2},$$

*and*

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}(1-|\Theta|^2)\right|\le C(\frac{\log\lambda}{\lambda^2})^2.$$

**Proof.** First equation is easy: in fact, by Eq. (7.8), we have

$$1-|\Theta|^2=2(4+o(1))\frac{\log\lambda}{\lambda^2}.$$

To get the second estimate, remember $P\cdot\frac{\mathrm{d}P}{\mathrm{d}t}=0$. Thus it easily follows from the estimate (7.8) that

$$\frac{\mathrm{d}}{\mathrm{d}t}(1-|\Theta|^2)=-2\Theta\cdot\frac{\mathrm{d}\Theta}{\mathrm{d}t}=-16\frac{\lambda_t}{\lambda}(\frac{\log\lambda}{\lambda^2}+O(\frac{1}{\lambda^2}))\tag{7.11}$$

Then we recall the equation just before the estimate (4.9):

$$\frac{\lambda_t}{\lambda}P+\frac{\lambda^2-1}{2\lambda}\frac{\mathrm{d}P}{\mathrm{d}t}=2\left(I-(\int_{\mathbb{S}^2}x^ix^jV^{-1}e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})\right)^{-1}b.$$

As we have noticed before, the inverse matrix $\left(VI-(\int_{\mathbb{S}^2}x^ix^je^{2v}\mathrm{d}\mu_{\mathbb{S}^2})\right)^{-1}$ is uniformly bounded.

Therefore, using $0 = \langle P, \frac{dP}{dt}\rangle$ and the estimates in Lemma 7.5, we have

$$\begin{aligned}\left|\frac{\lambda_t}{\lambda}\right| \leq & 2|\left(I - (\int_{\mathbb{S}^2} x^i x^j V^{-1} e^{2v} d\mu_{\mathbb{S}^2})\right)^{-1} b| \\ \leq & C\frac{\log\lambda}{\lambda^2}(|\Delta_{\mathbb{S}^2} f(\Theta)| + O(1)|\nabla f(\Theta)|^2_{\mathbb{S}^2} + o(1)).\end{aligned}$$

Inserting it into the identity (7.11), the second estimate follows easily. This completes the proof. □

The main purpose of the current section is to show the following lemma which will be useful in the next section for Moser theory argument.

**Lemma 7.7.** *The metric $g(t)$ concentrates at one of the critical points $Q$ of $f$ with $\Delta_{\mathbb{S}^2} f(Q) \leq 0$ as $t \to \infty$. Furthermore, the energy converges to a constant $-\log f(Q)$, that is, we have, as $t \to \infty$,*

$$E_f[u(t)] \to -\log f(Q).$$

**Proof.** By Lemma 7.6, we have

$$\left|\frac{d}{dt}(1 - |\Theta|^2)\right| \leq C(1 - |\Theta|^2)^2$$

Then for $t > t_0$ with $t_0$ sufficiently large,

$$1 - |\Theta(t)|^2 \geq \frac{C_0}{1+t}$$

with some uniform constant $C_0 > 0$.

By the first identity on Lemma 7.6, above estimate also implies

$$\frac{\log\lambda}{\lambda^2} \geq \frac{C_0}{1+t}.$$

This implies that

$$\frac{1}{\lambda^2} \leq \frac{C_1}{t\log t}. \tag{7.12}$$

Recall that Eq. (7.8) implies that the radial projection of $\Theta$ on $\mathbb{S}^2$ is just $P$. Thus we have

$$\begin{aligned}\frac{df(\Theta(t))}{dt} =& \frac{df(P(t))}{dt} \\ =& \frac{1}{|\Theta|}\langle\nabla f(P(t)), \frac{d\Theta}{dt}\rangle\end{aligned}$$

$$=\langle \nabla f(P(t)), \frac{\mathrm{d}P}{\mathrm{d}t}\rangle.$$

Take a look again at the equation just before the estimate (4.9). We simply rewrite it as the following:

$$\frac{\lambda^2-1}{2\lambda}\frac{\mathrm{d}P}{\mathrm{d}t} = 2(I-\Lambda_1)^{-1}b - 2([(I-\Lambda_1)^{-1}b]\cdot P)P, \tag{7.13}$$

where, on the right hand side, $\Lambda_1 = V^{-1}\Lambda$. Let us define $\Lambda_2 = \frac{3}{2}\int_{\mathbb{S}^2} x \cdot x^\top (V^{-1}e^{2v} - (4\pi)^{-1})\mathrm{d}\mu_{\mathbb{S}^2}$, where $x$ has been considered as a column vector so that $x \cdot x^T$ is a 3 by 3 matrix. Then we have the relation $\frac{2}{3}(I-\Lambda_2) = I - \Lambda_1$.

Notice that $(I-\Lambda_1)^{-1} = \frac{3}{2}(I + (I-\Lambda_2)^{-1}\Lambda_2)$, we obtain

$$\frac{\lambda^2-1}{2\lambda}\frac{\mathrm{d}P}{\mathrm{d}t} = 3(b-(b\cdot P)P) + 3(I-\Lambda_2)^{-1}\Lambda_2 b - 3([(I-\Lambda_2)^{-1}\Lambda_2 b]\cdot P)P.$$

Now apply Lemma 7.5 to get

$$\frac{\mathrm{d}f(\Theta(t))}{\mathrm{d}t} = \frac{6A_1\alpha}{f(P)V(\lambda^2-1)}|\nabla f(P)|^2_{\mathbb{S}^2} + o(\frac{1}{\lambda^2}).$$

In this estimate, we have observed that first error term in $b-(b\cdot P)P$ is $o(1)$ since the error term is of order $\lambda^{-2}\log\lambda$ while the order of $\frac{\mathrm{d}P}{\mathrm{d}t}$ is $\frac{1}{\lambda}$. For reminder terms, we simply notice that $|\Lambda_2 b| \le C\|v-\tilde{v}_\infty\|_{H^2(\mathbb{S}^2)}|b|$. Now $|b|$ has order $\lambda^{-2}\log\lambda$ while $\|v-\tilde{v}_\infty\|_{H^2(\mathbb{S}^2)}$ has the same order, hence above equation follows.

Since the integral of $(t\log t)^{-1}$ is divergent, the flow $\{\Theta(t); t\ge 0\}$ accumulates at a critical point $Q$ of $f$.

Once we have known that $Q$ is a critical point of $f$, then Eq. (7.10) implies that $\|f_\varphi - f(\Theta)\|^2_{L^2(\mathbb{S}^2)} = o(1)\lambda^{-2}\log\lambda$. Note that $|b|^2 = o(1)\lambda^{-2}\log\lambda$. Thus we conclude that the remainder term in Eq. (7.9) is $o(1)\lambda^{-2}\log\lambda$. This implies that

$$b\cdot P = -A_2\frac{\alpha\log\lambda}{V\lambda^2}(\Delta_{\mathbb{S}^2}f(Q)+o(1)).$$

Thus, with the help of equation just before the estimate (4.9), we have

$$\begin{aligned}
\frac{\lambda_t}{\lambda} =& \frac{\lambda_t}{\lambda}P\cdot P\\
=&[2\left(I - (\int_{\mathbb{S}^2} x^i x^j V^{-1}e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})\right)^{-1} b]\cdot P\\
=&2[(I-\Lambda_1))^{-1}b]\cdot P\\
=&3[(I-\Lambda_2))^{-1}b]\cdot P\\
=&3[b+(I-\Lambda_2))^{-1}\Lambda_2 b]\cdot P.
\end{aligned}$$

As before the last term here is $o(1)\lambda^{-2}\log\lambda$, put into Eq. (7.11) to get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(1-|\Theta|^2)=&-16\frac{\lambda_t}{\lambda}(\frac{\log\lambda}{\lambda^2}+o(1)\frac{1}{\lambda^2})\\ =&48A_2\frac{\alpha}{V}(\frac{\log\lambda}{\lambda^2})^2[(\Delta_{\mathbb{S}^2}f)(Q)+o(1)][1+o(1)].\end{aligned}$$

It follows from this, if $\Delta_{\mathbb{S}^2}f(Q)>0$, then for $t$ sufficiently large, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1-|\Theta|^2\right)>0$$

for sufficiently large $t$, contradicting the fact that $|\Theta(t)|\to 1$ as $t\to\infty$. This implies that $\Delta_{\mathbb{S}^2}f(Q)\le 0$. Moreover, by Theorem 6.4 and conformal invariance of $E[u]$, as $t\to\infty$ we obtain

$$E[u(t)]=E[v(t)]\to 2v_\infty=-\log f(Q)$$

as desired. □

## 8. Existence of conformal metrics

Given $u_0\in C_*^\infty:=\{u\in C^\infty(\mathbb{S}^2);\ \fint_{\mathbb{S}^2}fe^{2u}\mathrm{d}\mu_{\mathbb{S}^2}=1\}$, let $u(t,u_0)$ be the solution of the flow (2.2) with an initial datum $u(0)=u_0$, where $t\ge 0$. For $P\in\mathbb{S}^2$, $1<\lambda<\infty$, we denote by $\varphi_{P,\lambda}$ the conformal transformation on $\mathbb{S}^2$. As we have seen before that $\varphi_{P,\lambda}$ converges weakly in $H^1(\mathbb{S}^2)$ to $P$ as $\lambda\to\infty$. Define a map

$$\mathbb{S}^2\times(0,\infty)\ni(P,\lambda)\mapsto u_{P,\lambda}=\frac{1}{2}\log\det(\mathrm{d}\varphi_{P,\lambda}).$$

If we set $g_{P,\lambda}=\varphi_{P,\lambda}^*(g_{\mathbb{S}^2})=e^{2u_{P,\lambda}}g_{\mathbb{S}^2}$, then

$$\mathrm{d}\mu_{g_{P,\lambda}}=e^{2u_{P,\lambda}}\mathrm{d}\mu_{\mathbb{S}^2}\rightharpoonup 4\pi\delta_{-P},$$

in the sense of measures as $\lambda\to\infty$. For convenience, we define

$$E_f[u]:=E[u]-\log\fint_{\mathbb{S}^2}fe^{2u}\mathrm{d}\mu_{\mathbb{S}^2}.$$

**Remark 8.1.** We give a way to construct the set $C_*^\infty$ of initial data. To this end, let $P$ be a maximum point of $f$ with $f(P)>0$ and

$$u_{-P,\lambda}=\frac{1}{2}\log\det\mathrm{d}\varphi_{-P,\lambda}\ \text{ for all }\ \lambda\gg 1.$$

Then we select $a_{P,\lambda}\in\mathbb{R}$ such that

$$e^{2a_{P,\lambda}} \int_{\mathbb{S}^2} f \circ \varphi_{-P,\lambda} \mathrm{d}\mu_{\mathbb{S}^2} = 4\pi.$$

Now we choose

$$u_0 = u_{-P,\lambda} + a_{P,\lambda},$$

then it is not hard to show that $u_0 \in C_*^\infty$.

For $\beta \in \mathbb{R}$, denote by

$$L_\beta = \{u \in C_*^\infty; E_f[u] \le \beta\}$$

the sublevel sets of $E_f[u]$. We label critical points $Q_1, Q_2, \cdots, Q_n$ of $f$ with positive critical values $f(Q_i) > 0$ in the way that $f(Q_i) \le f(Q_j)$ for $1 \le i \le j \le n$. For each $1 \le i \le n$, there holds $\beta_i = -\log f(Q_i) = \lim_{\lambda\to\infty} E_f[u_{-Q_i,\lambda}]$. For the notational convenience only, in the followings, we assume that all critical values $f(Q_i), 1 \le i \le n$ are distinct, then there exists a positive constant $\nu_0$ such that $\beta_i - \nu_0 > \beta_{i+1}$ for all $1 \le i \le n-1$.

**Proposition 8.2.** *Let* $\mathrm{ind}(f, Q_i)$ *denote the Morse index of* $f$ *at* $Q_i$*, then*

*(i.) For any* $\beta_0 > \beta_1$ *the set* $L_{\beta_0}$ *is contractible.*
*(ii.) For any* $0 < \nu \le \nu_0$ *the set* $L_{\beta_i - \nu}$ *is homotopy equivalent to the set* $L_{\beta_{i+1}+\nu}$.
*(iii.) For each critical point* $Q_i$ *of* $f$ *with* $\Delta_{\mathbb{S}^2} f(Q_i) > 0$ *the set* $L_{\beta_i+\nu_0}$ *is homotopy equivalent to the set* $L_{\beta_i-\nu_0}$.
*(iv.) For each point* $Q_i$ *of* $f$ *with* $\Delta_{\mathbb{S}^2} f(Q_i) < 0$ *the set* $L_{\beta_i+\nu_0}$ *is homotopy equivalent to the set* $L_{\beta_i-\nu_0}$ *with a sphere of dimension* $2 - \mathrm{ind}(f, Q_i)$ *attached.*

Assuming Proposition 8.2 holds true, we are now in a position to establish our main theorem.

**Proof of Theorem 1.1.** Suppose $f$ cannot be realized as the Gaussian curvature of a conformal metric of $g_{\mathbb{S}^2}$. As we shall explain, the flow (2.2) then may be useful to show that for suitably large $\beta_0$, the set $L_{\beta_0}$ is contractible. Moreover, the flow defines a homotopy equivalence of a set $N_0$ with $N_\infty$ whose homotopy type is that of a point $\{Q_0\}$ with cells of dimension $2 - \mathrm{ind}(f, Q)$ attached for every critical point $Q$ of $f$ on $\mathbb{S}^2$ such that $\Delta_{\mathbb{S}^2} f(Q) < 0$ and $f(Q) > 0$. We then, by standard Moser theory (see [7, Theorem 4.3 on p.36]), obtain the identity

$$\sum_{i=0}^{2} t^i m_i = 1 + (1+t) \sum_{i=0}^{2} t^i k_i,$$

where

$$m_i = \sharp\{Q \in \mathbb{S}^2; f(Q) > 0, \nabla f(Q) = 0, \Delta_{\mathbb{S}^2} f(Q) < 0, \mathrm{ind}(f, Q) = 2 - i\}.$$

Equating the coefficients in the polynomial on the left and right hand side we obtain a contradiction with condition $(c)$ by observation that $m_0 = p, m_1 = q$. This finishes the proof of our main theorem. $\square$

Due to the length of the proof of Proposition 8.2, we divide the proof into several lemmas. First lemma settles the case $(i.)$ in the Proposition 8.2..

**Lemma 8.3.** *For any $\beta_0 > \beta_1$, the sub-level set $L_{\beta_0}$ is contractible.*

**Proof.** Fix $\beta_0 > \beta_1 = -\log f(Q_1), u_0 \in L_{\beta_0}$, let

$$H(s,u_0) = \frac{1}{2}\log\left((1-s)e^{2u_0} + \frac{s}{\fint_{\mathbb{S}^2} f \mathrm{d}\mu_{\mathbb{S}^2}}\right), \quad s \in [0,1],$$

which implies that $\fint_{\mathbb{S}^2} f e^{2H(s,u_0)}\mathrm{d}\mu_{\mathbb{S}^2} = 1$, then $H(s,u_0) \in C_*^\infty$, that is, the map $H$ defines a contraction of $L_{\beta_0}$ within $C_*^\infty$. Notice that $u(t, H(s,u_0)) \in C_*^\infty$ for any $t \geq 0$ and $E_f[u(t, H(s,u_0))] \to -\log f(Q_i)$ as $t \to \infty$ by virtue of Lemma 7.7, then there exists a minimal $T(s,u_0) \geq 0$ such that

$$E_f[u(T(s,u_0), H(s,u_0))] \leq \beta_0.$$

By Lemma 2.1 we conclude that $T(s,u_0)$ continuously depends on $u_0$ and $s$. Hence, the following map

$$(s,u_0) \mapsto u(T(s,u_0), H(s,u_0))$$

yields the desired contradiction within $L_{\beta_0}$. This completes the argument. $\square$

Next lemma deals with the second part of Proposition 8.2.

**Lemma 8.4.** *For any $0 < \nu \leq \nu_0$ the set $L_{\beta_i - \nu}$ is homotopy equivalent to the set $L_{\beta_{i+1}+\nu}$.*

**Proof.** Given $0 < \nu \leq \nu_0$, we claim that there exists $T > 0$ such that $u(T, L_{\beta_i-\nu}) \subset L_{\beta_{i+1}+\nu}$. By negation, there exist $T_l \to \infty$ and $u_l \in L_{\beta_i - \nu}$ such that

$$E_f[u(T_l, u_l)] > \beta_{i+1} + \nu \quad \text{for all} \quad l \in \mathbb{N}.$$

By Lemma 5.1, there exist $t_l \in [T_l/2, T_l]$ such that

$$\int_{\mathbb{S}^2} (\alpha(t_l) f - K_l)^2 \mathrm{d}\mu_{g_l} \to 0 \qquad \text{as} \quad l \to \infty,$$

where $g_l = e^{2u(t_l,u_l)} g_{\mathbb{S}^2}$ and $K_l$ is the Gaussian curvature of $g_l$. For $l \gg 1$ the metric $g_l$ will concentrate at the points $Q(t_l, u_l) \in \mathbb{S}^2$ and

$$E_f[u(t_l,u_l)] + \log f(Q(t_l,u_l) \to 0 \qquad \text{as} \quad l \to \infty.$$

Up to a subsequence, we may assume $Q = \lim_{l\to\infty} Q(t_l,u_l)$ with $f(Q) > 0$ and conclude that

$$E_f[u(t_l,u_l)] \to -\log f(Q) \qquad \text{as } l \to \infty.$$

Since $E_f[u_l] \le \beta_i - \nu$ and Lemma 2.1, we have $Q = Q_{i_0}$ for some $i_0 > i$

$$E_f[u(T_l, u_l)] \le E_f[u(t_l, u_l)] \le \beta_{i+1} + \nu$$

for large $l$. This yields a contradiction.

For $u_0 \in L_{\beta_i - \nu}$, let

$$T(u_0) = \inf\{t \ge 0; E_f[u(t, u_0)] \le \beta_{i+1} + \nu\} \le T.$$

As in Lemma 8.3, $T(u_0)$ continuously depends on $u_0$. The map

$$(t, u_0) \mapsto u(\min\{t, T(u_0)\}, u_0)$$

yields the desired homotopy equivalence. $\square$

In order to handle third and fourth parts, we have to do some technical preparation.

**Lemma 8.5.** *With a uniform constant $C > 0$, there holds*

$$\|\hat{v}\|^2_{H^1(\mathbb{S}^2)} \le C E[\hat{v}]$$

*for all $\hat{v} \in H^2(\mathbb{S}^2)$ satisfying*

$$\fint_{\mathbb{S}^2} e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2} = 1 \quad \text{and} \quad \fint_{\mathbb{S}^2} x e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2} = 0,$$

*provided that $\|\hat{v}\|_{H^1(\mathbb{S}^2)}$ is sufficiently small.*

**Proof.** By spectrum decomposition for $e^{2\hat{v}} - 1$ and $v$, one has

$$e^{2\hat{v}} - 1 = \sum_{i=0}^{\infty} V^i \varphi_i\,, \quad \hat{v} = \sum_{i=0}^{\infty} v^i \varphi_i.$$

Take the Taylor expansion on $e^{2\hat{v}} - 1$ to get the relation between $V^i$ and $v^i$ as follows:

$$V^i = 2v^i + O(\|\hat{v}\|^2_{H^1(\mathbb{S}^2)}) \quad \text{for all} \quad i \in \mathbb{N}.$$

It follows that $|v^i|^2 = o(\|\hat{v}\|^2_{H^1(\mathbb{S}^2)})$ for $0 \le i \le 3$. Thus, we obtain

$$\begin{aligned} E[\hat{v}] &= \fint_{\mathbb{S}^2} (|\nabla \hat{v}|^2_{\mathbb{S}^2} + 2\hat{v}) \mathrm{d}\mu_{\mathbb{S}^2} \\ &= \fint_{\mathbb{S}^2} [|\nabla \hat{v}|^2_{\mathbb{S}^2} + e^{2\hat{v}} - 1 - 2\hat{v}^2 + O(|\hat{v}|^3)] \mathrm{d}\mu_{\mathbb{S}^2} \end{aligned}$$

$$
\begin{aligned}
&=\sum_{i=0}^{\infty}(\lambda_i-2)|v^i|^2+o(\|\hat{v}\|^2_{H^1(\mathbb{S}^2)})\\
&\geq\frac{\lambda_4-2}{\lambda_4+1}\sum_{i\geq 0}(\lambda_i+1)|v^i|^2+o(\|\hat{v}\|^2_{H^1(\mathbb{S}^2)})\\
&\geq C\,\|\hat{v}\|^2_{H^1(\mathbb{S}^2)},
\end{aligned}
$$

where we have used the estimate

$$\int_{\mathbb{S}^2}|\hat{v}|^3 \mathrm{d}\mu_{\mathbb{S}^2}\leq\|\hat{v}\|^2_{L^4(\mathbb{S}^2)}\|\hat{v}\|_{L^2(\mathbb{S}^2)}\leq C\|\hat{v}\|^2_{H^1(\mathbb{S}^2)}\|\hat{v}\|^2_{L^2(\mathbb{S}^2)}=o(1)\|\hat{v}\|^2_{H^1(\mathbb{S}^2)}$$

by Hölder's inequality and Sobolev inequality (A.1). □

Before we state the next Lemma, we set some notations first.

For simplicity, we set $\hat{v}=v-\tilde{v}_\infty$ with $\tilde{v}_\infty=(\log \fint_{\mathbb{S}^2}e^{2v}\mathrm{d}\mu_{\mathbb{S}^2})/2$, $\theta=s\sqrt{|\log s|}$ for any $s\in\mathbb{R}_+$, and we define

$$
\begin{aligned}
B_{r_0}(Q_i):=\Big\{u\in C^\infty_*;\ \ &g=e^{2u}g_{\mathbb{S}^2}\ \text{ induces a normalized metric}\\
&h=\varphi^*_{-P,s}g=e^{2v}g_{\mathbb{S}^2}\ \text{for some } P\in\mathbb{S}^2 \text{ and } 0<s\leq e^{-1}\\
&\text{such that}\ \ \|\hat{v}\|^2_{H^1(\mathbb{S}^2)}+|P-Q_i|^2+\theta^2<r_0^2\Big\}
\end{aligned}
$$

for $r_0\in\mathbb{R}_+$, $1\leq i\leq n$ and $s=\lambda^{-1}$.

For convenience, we use $(\theta,P,\hat{v})$ as coordinates for $u\in B_{r_0}(Q_i)$. Since all critical points of $f$ by assumption are non-degenerate, the Morse lemma shows that there exist local coordinates such that $P=P^++P^-$ on $T_P(\mathbb{S}^2)$ near $Q_i=0$, and, by Morse coordinates, near $Q_i$,

$$f(P)=f(Q_i)+|P^+|^2-|P^-|^2.$$

Now we state our next technical lemma which mainly concerns with the expansions of involved quantities in terms of the coordinates $(\theta,P,\hat{v})$.

**Lemma 8.6.** *Let $u=(\theta,P,\hat{v})\in B_{r_0}(Q_i)$ for some $1\leq i\leq n$ and sufficiently small $r_0>0$, then*
*(a)*

$$A:=\fint_{\mathbb{S}^2}f\circ\varphi_{-P,s}e^{2\hat{v}}\mathrm{d}\mu_{\mathbb{S}^2}=f(P)+2\theta^2\Delta_{\mathbb{S}^2}f(P)+o(1)(\theta^2+\theta\,\|\hat{v}\|_{H^1(\mathbb{S}^2)}).$$

*(b)*

$$\Big|\partial_\theta E_f[u]+4A^{-1}\theta\Delta_{\mathbb{S}^2}f(P)\Big|=o(1)(\theta+\|\hat{v}\|_{H^1(\mathbb{S}^2)}).$$

*(c) For any $q\in T_P\mathbb{S}^2$, there holds*

$$\Big|\partial_P E_f[u]\cdot q+A^{-1}\mathrm{d}f(P)\cdot q\Big|\leq C\theta(\theta+\|\hat{v}\|_{H^1(\mathbb{S}^2)})|q|.$$

(d) *Denoting by* $\langle\cdot,\cdot\rangle$ *the duality pairing of* $H^1(\mathbb{S}^2)$ *with its dual space* $H^{-1}(\mathbb{S}^2)$. *Then there exists a uniform constant* $c_0 > 0$ *such that*

$$\langle \partial_{\hat{v}} E_f[u], \hat{v}\rangle \geq c_0 \left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)} - o(1)\theta \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)},$$

*with* $o(1) \to 0$ *as* $r_0 \to 0$.

**Proof. Step (a):** Since $\fint_{\mathbb{S}^2} e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2} = 1$, we have

$$A = f(P) + \fint_{\mathbb{S}^2} (f \circ \varphi_{-P,s} - f(P)) \mathrm{d}\mu_{\mathbb{S}^2} + I,$$

where

$$I = \fint_{\mathbb{S}^2} (f \circ \varphi_{-P,s} - f(P))(e^{2\hat{v}} - 1) \mathrm{d}\mu_{\mathbb{S}^2}.$$

In $B_{s^{-1}}(0)$, the Taylor expansion in local coordinates around $P$ gives

$$f \circ \varphi_{-P,s} - f(P) = s\mathrm{d}f(P)z + \frac{s^2}{2}\nabla \mathrm{d}f(P)(z,z) + O(s^3|z^3|). \tag{8.1}$$

With the above expansion, we estimate

$$\fint_{\mathbb{S}^2} (f \circ \varphi_{-P,s} - f(P)) \mathrm{d}\mu_{\mathbb{S}^2} = -s^2|\log s|\Delta_{\mathbb{S}^2} f(P) + O(s^2) = \theta^2 \Delta_{\mathbb{S}^2} f(P) + o(1)\theta^2$$

and the error term

$$\begin{aligned}
|I| &\leq \left\| f \circ \varphi_{-P,s} - f(P)\right\|_{L^2(\mathbb{S}^2)} \left\| e^{2\hat{v}} - 1\right\|_{L^2(\mathbb{S}^2)} \\
&\leq C(s\sqrt{|\log s|}|\nabla f(P)|_{\mathbb{S}^2} + s) \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)} \\
&\leq C\left(|P - Q_i| + \frac{1}{\sqrt{|\log s|}}\right)\theta \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)} \\
&= o(1)\theta \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)},
\end{aligned}$$

where the second inequality follows from

$$\begin{aligned}
\fint_{\mathbb{S}^2} (e^{2\hat{v}} - 1)^2 \mathrm{d}\mu_{\mathbb{S}^2} = \fint_{\mathbb{S}^2} (e^{4\hat{v}} - 1) \mathrm{d}\mu_{\mathbb{S}^2} &\leq \exp\left\{ 4 \fint_{\mathbb{S}^2} (|\nabla \hat{v}|^2_{\mathbb{S}^2} + \hat{v}) \mathrm{d}\mu_{\mathbb{S}^2} \right\} - 1 \\
&\leq e^{c_1 \left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}} - 1 \leq C \left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}
\end{aligned}$$

by virtue of Onofri-Moser-Trudinger's inequality. Therefore, the first assertion follows by collecting the above estimates together.

**Step (b):** Using

$$E_f[u] = E[\hat{v}] - \log \fint_{\mathbb{S}^2} f \circ \varphi_{-P,s} e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2},$$

we compute

$$\partial_\theta E_f[u] = -\frac{1}{A}\frac{\partial A}{\partial \theta}$$

and

$$\begin{aligned}
\frac{\partial A}{\partial \theta} =& \frac{\mathrm{d}s}{\mathrm{d}\theta}\left[\fint_{\mathbb{S}^2} \partial_s(f \circ \varphi_{-P,s})\mathrm{d}\mu_{\mathbb{S}^2} + \fint_{\mathbb{S}^2} \partial_s(f \circ \varphi_{-P,s})(e^{2\hat{v}} - 1)\mathrm{d}\mu_{\mathbb{S}^2}\right] \\
:=& \frac{\mathrm{d}s}{\mathrm{d}\theta}(II_1 + II_2).
\end{aligned}$$

By relation $\theta^2 = -s^2 \log s$, we have

$$\frac{\mathrm{d}s}{\mathrm{d}\theta} = \frac{2\theta}{-2s\log s - s} = \frac{2\sqrt{|\log s|}}{2|\log s| - 1}.$$

Eq. (8.1) yields

$$\begin{aligned}
II_1 =& \fint_{\mathbb{S}^2} \partial_s(f \circ \varphi_{-P,s})\mathrm{d}\mu_{\mathbb{S}^2} \\
=& \frac{1}{4\pi}\int_{B_{\frac{1}{s}}(0)} [\mathrm{d}f(P)z + s\nabla \mathrm{d}f(P)(z,z) + O(s^2|z|^3)]\frac{4\mathrm{d}z}{(1+|z|^2)^2} \\
&+ O(1)\int_{B^c_{1/s}(0)} |z|\frac{\mathrm{d}z}{(1+|z|^2)^2} \\
=& (\Delta_{\mathbb{S}^2} f)(P)s\int_0^{\frac{1}{s}} \frac{r^3}{(1+r^2)^2}\mathrm{d}r + O(s) \\
=& s|\log s|\Delta_{\mathbb{S}^2} f(P) + O(s),
\end{aligned}$$

with the observation that, on the outside region $B^c_{\frac{1}{s}}(0)$, $|\partial_s f(sz)| \le |z||\nabla f|(sz)| \le C|z|$.

Then, we have

$$\frac{\mathrm{d}s}{\mathrm{d}\theta}\, II_1 = \frac{2\theta}{2s|\log s| - s} \cdot s|\log s| \Delta_{\mathbb{S}^2} f(P) + O(1)\frac{\theta}{2|\log s| - 1}$$
$$= \theta \Delta_{\mathbb{S}^2} f(P) + o(1)\theta.$$

Notice that

$$|II_2| \le C \left\| \partial_s (f \circ \varphi_{-P,s}) \right\|_{L^2(\mathbb{S}^2)} \left\| e^{2\hat{v}} - 1 \right\|_{L^2(\mathbb{S}^2)}.$$

First with similar argument as in $II_1$, we have

$$\|\partial_s (f \circ \varphi_{-P,s})\|_{L^2(\mathbb{S}^2)} \le C(|P - Q_i|\sqrt{|\log s|} + 1),$$

where we have used the fact that $\nabla f(Q_i) = 0$.

Thus, with the help of the estimate on $\hat{v}$, we obtain

$$\left|\frac{\mathrm{d}s}{\mathrm{d}\theta}\right| \cdot |II_2| \le C(|P - Q_i| + \frac{1}{\sqrt{|\log s|}}) \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)} = o(1) \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)}.$$

Consequently, putting these facts together we obtain

$$\left|\partial_\theta E_f[u] + 4A^{-1}\theta \Delta_{\mathbb{S}^2} f(P)\right| = o(1)(\theta + \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)}).$$

**Step (c):** For any $q \in T_P\mathbb{S}^2$, we have

$$\begin{aligned}
& A\ \partial_P E_f[u] \cdot q + \mathrm{d}f(P) \cdot q \\
=& \fint_{\mathbb{S}^2} (\mathrm{d}f(P) - \partial_P (f \circ \varphi_{-P,s})) \cdot q e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2} \\
=& \fint_{\mathbb{S}^2} (\mathrm{d}f(P) - \partial_P (f \circ \varphi_{-P,s})) \cdot q \mathrm{d}\mu_{\mathbb{S}^2} \\
& + \fint_{\mathbb{S}^2} (\mathrm{d}f(P) - \partial_P (f \circ \varphi_{-P,s})) \cdot q (e^{2\hat{v}} - 1) \mathrm{d}\mu_{\mathbb{S}^2} \\
:=& III_1 + III_2.
\end{aligned}$$

By the expansion for $f$, i.e., Eq. (8.1) and rotational symmetry, we estimate

$$|III_1| \le C \left[ \int_{B_{\frac{1}{s}}(0)} \frac{s^2|z|^2 + O(s^3|z|^3)}{(1+|z|^2)^2} \mathrm{d}z + \int_{B^c_{1/s}(0)} \frac{\mathrm{d}z}{(1+|z|^2)^2} \right] |q| \le Cs^2|\log s||q| \le C\theta^2|q|$$

and

$$|III_2| \leq C|q| \left\| \mathrm{d}f(P) - \partial_P(f \circ \varphi_{-P,s}) \right\|_{L^2(\mathbb{S}^2)} \left\| e^{2\hat{v}} - 1 \right\|_{L^2(\mathbb{S}^2)} \leq C\theta \left\| \hat{v} \right\|_{H^1(\mathbb{S}^2)} |q|.$$

Hence the desired assertion follows from the above estimates.

**Step (d):** We decompose

$$\begin{aligned}\langle \partial_{\hat{v}} E_f[u], \hat{v} \rangle &= \langle \partial_{\hat{v}} E[\hat{v}], \hat{v} \rangle - 2A^{-1} \fint_{\mathbb{S}^2} f \circ \varphi_{-P,s} \hat{v} e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2} \\ &= 2 \fint_{\mathbb{S}^2} |\nabla \hat{v}|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2 \fint_{\mathbb{S}^2} \hat{v}(1 - e^{2\hat{v}}) \mathrm{d}\mu_{\mathbb{S}^2} - IV_1 - IV_2,\end{aligned}$$

where

$$IV_1 = 2A^{-1} \fint_{\mathbb{S}^2} (f \circ \varphi_{-P,s} - f(P)) \hat{v} e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2}$$

and

$$IV_2 = 2(A^{-1} f(P) - 1) \fint_{\mathbb{S}^2} \hat{v} e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2}.$$

As in the proof of Lemma 7.4 we have

$$\begin{aligned}&2 \fint_{\mathbb{S}^2} |\nabla \hat{v}|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + 2 \fint_{\mathbb{S}^2} \hat{v}(1 - e^{2\hat{v}}) \mathrm{d}\mu_{\mathbb{S}^2} \\ =&2 \fint_{\mathbb{S}^2} (|\nabla \hat{v}|^2_{\mathbb{S}^2} - 2\hat{v}^2) \mathrm{d}\mu_{\mathbb{S}^2} + o(1) \|\hat{v}\|^2_{H^1(\mathbb{S}^2)} \\ \geq&2 \frac{\lambda_4 - 2}{\lambda_4 + 1} \sum_{i=4}^{\infty} (\lambda_i + 1)|v^i|^2 + o(1) \|\hat{v}\|^2_{H^1(\mathbb{S}^2)} \\ =&\left( \frac{8}{7} + o(1) \right) \|\hat{v}\|^2_{H^1(\mathbb{S}^2)}.\end{aligned}$$

By the expansion (8.1) and our **Step (a)**, we can bound

$$|IV_1| \leq C \left\| f \circ \varphi_{-P,s} - f(P) \right\|_{L^2(\mathbb{S}^2)} \|\hat{v}(e^{2\hat{v}} - 1)\|_{L^2(\mathbb{S}^2)} \leq C\theta \left\| \hat{v} \right\|^2_{H^1(\mathbb{S}^2)}$$

and

$$|IV_2| \leq C|A - f(P)| \int_{\mathbb{S}^2} |\hat{v}| e^{2\hat{v}} \mathrm{d}\mu_{\mathbb{S}^2} \leq C\theta^2 \left\| \hat{v} \right\|_{H^1(\mathbb{S}^2)}.$$

Hence, we combine all above estimates to conclude the desired assertion. □

Before we start with the proof of Proposition 8.2 ($iii.$), we have to prepare some notations and formulas.

In the following we choose small enough $r_0, \nu > 0$ such that $B_{r_0}(Q_i) \subset L_{\beta_i+\nu_0} \setminus L_{\beta_i-\nu_0}$ and $\nu \le r_0^3 < \nu_0$. For any $1 \le i \le n$ and sufficiently large $T > 0$, it follows from *ii.* that $u(T, L_{\beta_i+\nu_0}) \subset L_{\beta_i+\nu}$. Moreover, choosing a larger number $T$, if necessary, given any $u_0 \in L_{\beta_i+\nu_0}$ we have either $u(T, u_0) \in L_{\beta_i-\nu}$ or $u(t, u_0) \in B_{r_0/2}(Q_i)$ for some $t \in [0, T]$.

Notice that

$$E_f[u] = E_f[u + c], \qquad c \in \mathbb{R}. \tag{8.2}$$

For each $u = (\theta, P, \hat{v}) \in B_{r_0}(Q_i)$, by definition of $B_{r_0}(Q_i)$, (8.2) and conformal invariance of $E[u]$ we have

$$E_f[u] = E_{f\circ\varphi_{-P,s}}[\hat{v}] = E[\hat{v}] + \beta_i - J, \tag{8.3}$$

where

$$J = \log\left(1 + \frac{A - f(Q_i)}{f(Q_i)}\right) = \frac{A - f(Q_i)}{f(Q_i)} + O(|A - f(Q_i)|^2).$$

By Lemma 8.6 *(a)*, we have

$$\begin{aligned} A - f(Q_i) &= A - f(P) + f(P) - f(Q_i) \\ &= 2\theta^2 \Delta_{\mathbb{S}^2} f(P) + |P^+|^2 - |P^-|^2 + o(1)(\theta^2 + \theta \left\| \hat{v} \right\|_{H^1(\mathbb{S}^2)}), \end{aligned}$$

which implies that

$$J \cdot f(Q_i) = 2\theta^2 \Delta_{\mathbb{S}^2} f(P) + |P^+|^2 - |P^-|^2 + o(1)(\theta^2 + |P - Q_i|^2 + \left\| \hat{v} \right\|^2_{H^1(\mathbb{S}^2)}). \tag{8.4}$$

With a uniform constant $c_1 > 0$, Eq. (8.4), together with Lemma 8.5, yields

$$E_f[u] \ge \beta_i + c_1 \left\| \hat{v} \right\|^2_{H^1(\mathbb{S}^2)} - C(\theta^2 + |P - Q_i|^2).$$

Thus, for $u \in L_{\beta_i+\nu} \cap B_{r_0}(Q_i)$ we have

$$\|\hat{v}\|^2_{H^1(\mathbb{S}^2)} \le C(\theta^2 + |P - Q_i|^2 + r_0^3). \tag{8.5}$$

On the other hand, with uniform constants $c_2, c_3 > 0$, Lemma 2.1, together with Lemma 7.5, yields

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t} E_f[u(t, u_0)] &\le -c_2 s^2 (|\nabla f(P(t))|^2_{\mathbb{S}^2} + s^2 |\log s|^2 |\Delta_{\mathbb{S}^2} f(P)|^2) \\ &\le -c_3 s^2 (|P(t) - Q_i|^2 + \theta^2). \end{aligned} \tag{8.6}$$

Hence, for $u_0 \in B_{r_0}(Q_i) \setminus B_{r_0/2}(Q_i)$, with a uniform constant $c_4 > 0$ and Eq. (8.5), we have

$$|P - Q_i|^2 + \theta^2 \geq c_4 r_0^2. \tag{8.7}$$

We now rescale time $t$ by solving

$$\frac{\mathrm{d}\tau}{\mathrm{d}t} = \min\{1/2, s(t, u_0)^2\}, \qquad \tau(0) = 0.$$

Keep in mind that $s^2 \geq C_1(t \log t)^{-1}$ by virtue of the estimate (7.12), for $t$ sufficiently large. This implies that $\tau(t) \to \infty$ as $t \to \infty$. We denote by $U(\tau, u_0)$ the flow $u(t(\tau), u_0)$ with an initial datum $u_0 \in C_*^\infty$. Then it follows from estimates (8.6) and (8.7) that with a uniform constant $c_5 > 0$

$$\frac{\mathrm{d}}{\mathrm{d}\tau} E_f[U(\tau, u_0)] \leq -c_5 r_0^2 \qquad \text{for any } u_0 \in B_{r_0}(Q_i) \setminus B_{r_0/2}(Q_i). \tag{8.8}$$

Therefore, we conclude that for any fixed $r_0$, the length of time $\tau$ needed for the flow $U(\tau, u_0)$ to travel out transversely the annular region $L_{\beta_i+\nu} \cap (B_{r_0}(Q_i) \setminus B_{r_0/2}(Q_i))$ is uniformly positive. For sufficiently large $T > 0$ and sufficiently small $\nu > 0$ we have

$$U(T, L_{\beta_i+\nu_0}) \subset L_{\beta_i-\nu} \cup (B_{\frac{r_0}{2}}(Q_i) \cap L_{\beta_i+\nu}).$$

We define

$$T_\nu(u_0) := \min\{T, \inf\{\tau \geq 0; E_f[U(\tau, u_0)] \leq \beta_i - \nu\}\}$$

which continuously depends on $u_0$ as can be seen easily. Then the map $(t, u_0) \mapsto U(\min\{t, T_\nu(u_0)\}, u_0)$ defines a homotopy equivalence of $L_{\beta_i+\nu_0}$ with a subset of $L_{\beta_i-\nu} \cup (B_{r_0/2}(Q_i) \cap L_{\beta_i+\nu})$.

A simple but vital observation is that if $u = (\theta, P, \hat{v}) \in B_{r_0}(Q_i)$ with $⨍_{\mathbb{S}^2} f e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} > 0$, and we define

$$\|u\|^2 := \theta^2 + |P - Q_i|^2 + \|\hat{v}\|^2_{H^1(\mathbb{S}^2)},$$

then $\tilde{u} := u + c(u) \in C_*^\infty$ satisfies $\|\tilde{u}\| = \|u\|$, where $c(u) = -(⨍_{\mathbb{S}^2} f e^{2u} \mathrm{d}\mu_{\mathbb{S}^2})/2$. To that end, it suffices to show that

$$v = u \circ \varphi_{-P,s} + \frac{1}{2} \log \det \mathrm{d}\varphi_{-P,s} \quad \text{and} \quad v_1 = v + c(v),$$

where $v_1$ is the normalized conformal factor corresponding to $\tilde{u}$, then, together with the fact that the area of metric is also conformally invariant,

$$\begin{aligned}\hat{v}_1 &= v_1 - \frac{1}{2}\log \fint_{\mathbb{S}^2} e^{2v_1}\mathrm{d}\mu_{\mathbb{S}^2}\\ &= v + c(v) - \frac{1}{2}\log \fint_{\mathbb{S}^2} e^{2(v+c(v))}\mathrm{d}\mu_{\mathbb{S}^2}\\ &= v - \frac{1}{2}\log \fint_{\mathbb{S}^2} e^{2v}\mathrm{d}\mu_{\mathbb{S}^2} = \hat{v}.\end{aligned}$$

Indeed, $\|\cdot\|$ is invariant under any constant translation.

According to the sign of $\Delta_{\mathbb{S}^2} f(Q_i)$ we can proceed as follows.

**Lemma 8.7.** *For each critical point $Q_i$ of $f$ with $\Delta_{\mathbb{S}^2} f(Q_i) > 0$ the set $L_{\beta_i+\nu_0}$ is homotopy equivalent to the set $L_{\beta_i-\nu_0}$.*

**Proof.** Suppose $\Delta_{\mathbb{S}^2} f(Q_i) > 0$. For each $u = (\theta, P, \hat{v}) \in B_{r_0}(Q_i)$, we can choose $r_0 > 0$ sufficiently small such that

$$A = V^{-1}\fint_{\mathbb{S}^2} f e^{2u}\mathrm{d}\mu_{\mathbb{S}^2} = \fint_{\mathbb{S}^2} f\circ\varphi_{-P,s}e^{2\hat{v}}\mathrm{d}\mu_{\mathbb{S}^2} > 0.$$

Let $X(u)$ be a vector field defined as

$$X(u) = (1, 0, 0)$$

and $G(u, r)$ solve the flow equation

$$\frac{\mathrm{d}}{\mathrm{d}r}G(u,r) = X(G(u,r)), \quad 0 \le r \le r_0; \quad G(u,0) = u.$$

Notice that $\tilde{G}(u,r) = G(u,r) + c(G) \in C_*^\infty$ and $X$ is transversal to $\partial B_{r_0}(Q_i)$ such that $\tilde{G}(u,r_0) \notin B_{r_0}(Q_i)$, then there exists a first time $0 \le r = r(u) \le r_0$ such that $\tilde{G}(u,r) \notin B_{r_0}(Q_i)$ and the map $u \mapsto r(u)$ is continuous. Defining $H(u,r) := \tilde{G}(u, \min\{r, r(u)\})$ we obtain a homotopy

$$H : \overline{B_{r_0}(Q_i)} \times [0, r_0] \to \overline{B_{r_0}(Q_i)}$$

such that

$$H(B_{r_0}(Q_i), r_0) \subset \partial B_{r_0}(Q_i); \quad H(\cdot, r)|_{\partial B_r(Q_i)} = \mathrm{id}, \quad 0 \le r \le r_0.$$

For brevity, we set $u_r = G(u, \min\{r, r(u)\})$ for $0 \le r \le r(u)$. Then by Eq. (8.2), Lemmas 8.3 and 8.4, we have

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}E_f[H(u,r)] =& \frac{\mathrm{d}}{\mathrm{d}r}E_f[u_r] \\
=& \mathrm{d}E_f[u_r]\cdot X(u_r) \\
=& \partial_\theta E_f[u_r] \\
\le& -4A^{-1}\theta\Delta_{\mathbb{S}^2}f(P)+o(r_0) \\
\le& -4\theta f(P)^{-1}\Delta_{\mathbb{S}^2}f(P)+o(r_0).
\end{aligned}
$$

This implies that there exists a uniform constant $c_1 > 0$ such that

$$E_f[H(u,r_0)] \le E_f[u]-c_1r_0^2 < \beta_i-\nu$$

for all $u \in B_{\frac{r_0}{2}}(Q_i)\cap L_{\beta_i+\nu}$, provided that $r_0 > 0$ is sufficiently small.

Composing $H$ with the flow $(t,u_0)\mapsto U(\min\{t,T_\nu(u_0)\},u_0)$ we obtain a homotopy

$$K : L_{\beta_i+\nu_0}\times[0,1]\to L_{\beta_i+\nu_0}$$

such that $K(L_{\beta_i+\nu_0},1)\subset L_{\beta_i-\nu}$. Moreover, by our choice of $r_0>0$, then for $0\le r\le 1$, the map $K(\cdot,r)$ will be the identity map on $L_{\beta_i-\nu_0}$. For each $u_0\in L_{\beta_i-\nu}$, set $T_0(u_0)=\inf\{t\ge 0; E_f[u(t,u_0)]\le\beta_i-\nu_0\}$. Then $T_0(u_0)$ are uniformly bounded and continuously depend on $u_0$. This defines a desired homotopy equivalence. $\square$

Then we just have one more claim left and restate it here.

**Lemma 8.8.** *For each point $Q_i$ of $f$ with $\Delta_{\mathbb{S}^2}f(Q_i)<0$ the set $L_{\beta_i+\nu_0}$ is homotopy equivalent to the set $L_{\beta_i-\nu_0}$ with a sphere of dimension $2-\mathrm{ind}(f,Q_i)$ attached.*

**Proof.** By Eqs. (8.3) and (8.4), we obtain

$$
\begin{aligned}
E_f[u]-\beta_i \ge& c_1\|\hat{v}\|^2_{H^1(\mathbb{S}^2)}-2\theta^2f(Q_i)^{-1}\Delta_{\mathbb{S}^2}f(P)+f(Q_i)^{-1}(|P^-|^2-|P^+|^2) \\
&+o(1)(\theta^2+|P-Q_i|^2+\left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}),
\end{aligned}
$$

where $o(1)\to 0$ as $r_0\to 0$.

Suppose $\Delta_{\mathbb{S}^2}f(Q_i)<0$. Then there exists a real number $\delta>0$ such that

$$\theta^2+|P^-|^2+\left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}<\frac{r_0^2}{4} \tag{8.9}$$

for any $u=(\theta,P,\hat{v})\in B_{r_0}(Q_i)\cap L_{\beta_i+\nu}$ with $|P^+|<2\delta r_0$, provided that $r_0>0$ and $\nu\le r_0^3$ are sufficiently small. Let $\eta$ be a cut-off function given by

$$\eta(P)=\left(1-\frac{(|P^+|-\delta r_0)_+}{\delta r_0}\right)_+,$$

where $a_+=\max\{a,0\}$ for $a\in\mathbb{R}$. For convenience, we fix some $\theta_0\in(0,r_0^3)$ and define

$$B_\rho^+ := \{u \in B_{r_0}(Q_i); \theta = \theta_0, P^- = 0, |P^+| < \rho, \hat{v} = 0\}$$

for $0 < \rho < r_0$. For $0 \le r \le 1$, $u \in B_{r_0}(Q_i)$, we define $H_0(u, r) := \tilde{u}_r = u_r + c(u_r) \in C_*^\infty$ with

$$u_r = (\theta_r, P_r, \hat{v}_r) = (r\eta\theta_0 + (1 - r\eta)\theta, P^+ + (1 - r\eta)P^-, (1 - r\eta)\hat{v}).$$

We claim that, with suitably small constants $r_0, \delta > 0$, $H_0$ maps $\overline{B_{r_0}(Q_i)}$ to itself. To that end, we have

$$\begin{aligned}\|\tilde{u}_r\|^2 =& \|u_r\|^2 = \theta_r^2 + |P_r|^2 + \left\|\hat{v}_r\right\|_{H^1(\mathbb{S}^2)}^2 \\ =& r^2\eta^2\theta_0^2 + (1 - r\eta)^2(\theta^2 + |P^-|^2 + \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)}^2) + |P^+|^2 + 2r\eta\theta_0(1 - r\eta)\theta. \quad (8.10)\end{aligned}$$

If $|P^+| \ge 2\delta r_0$, then $\eta = 0$ and Eq. (8.10) becomes

$$\|\tilde{u}_r\|^2 = \theta^2 + |P^-|^2 + |P^+|^2 + \left\|\hat{v}\right\|_{H^1(\mathbb{S}^2)}^2 < r_0^2.$$

If $|P^+| < 2\delta r_0$ and let $s := r\eta \in [0, 1]$ for brevity, then by the inequality (8.9)

$$\text{RHS of Eq. (8.10)} < \frac{r_0^2}{4}(1 - s)^2 + s^2\theta_0^2 + 4\delta^2 r_0^2 + 2s(1 - s)\theta\theta_0 := F(s).$$

We rewrite

$$F(s) = \left(\frac{r_0^2}{4} + \theta_0^2 - 2\theta\theta_0\right)s^2 + \left(2\theta\theta_0 - \frac{r_0^2}{2}\right)s + \frac{r_0^2}{4} + 4\delta^2 r_0^2$$

and again by the inequality (8.9)

$$\frac{r_0^2}{4} + \theta_0^2 - 2\theta\theta_0 > \theta^2 + \theta_0^2 - 2\theta\theta_0 \ge 0.$$

This implies that $F(s) \le \max\{F(0), F(1)\}$ with

$$\begin{aligned}F(0) =& \frac{r_0^2}{4} + 4\delta^2 r_0^2 < r_0^2, \\ F(1) =& \theta_0^2 + 4\delta^2 r_0^2 < r_0^6 + \frac{3}{4}r_0^2 < r_0^2\end{aligned}$$

by choosing $r_0^4 < 1/4$ and $\delta^2 < 3/16$. Thus, the above claim follows.

Furthermore, $H_0$ gives a homotopy

$$H_0 : \overline{B_{r_0}(Q_i)} \cap L_{\beta_i+\nu} \times [0, 1] \to \overline{B_{r_0}(Q_i)},$$

where $H_0(\cdot, 1)$ maps the set $\{u \in B_{r_0}(Q_i) \cap L_{\beta_i+\nu}; |P^+| < \delta r_0\}$ to the set $B_{\delta r_0}^+$. Note that set $B_{\delta r_0}^+$ is diffeomorphic to a unit ball of dimension $2 - \operatorname{ind}(f, Q_i)$.

By Eq. (8.2), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}E_f[H_0(u,r)] =& \frac{\mathrm{d}}{\mathrm{d}r}E_f[u_r]\\ &=\eta\left(\partial_\theta E_f[u_r](\theta_0-\theta)-\partial_P E_f[u_r]\cdot P^- - \langle\partial_{\hat{v}}E_f[u_r],\hat{v}\rangle\right):=\eta D.\end{aligned}$$

The estimates in Lemma 8.6 provide the following control on $D$:

$$\begin{aligned}D \le& C\theta_0\theta+4f(P)^{-1}\theta^2\Delta_{\mathbb{S}^2}f(P)-2f(P)^{-1}|P^-|^2-c_0\left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}\\ &+o(1)(\theta^2+\left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}+|P^-|^2+\theta_0^2).\end{aligned}$$

If $r_0$ is sufficiently small, then the term $D$ is negative whenever $\theta^2+\left\|\hat{v}\right\|^2_{H^1(\mathbb{S}^2)}+|P^-|^2\ge r_0^4$; otherwise $E_f[u_r]\le\beta_i+Cr_0^4<\beta_i+\nu$ by virtue of Eqs. (8.3) and (8.4). Thus, for all $r$, $H_0(\cdot,r)$ maps the set $B_{r_0}(Q_i)\cap L_{\beta_i+\nu}$ into itself. Finally, by (8.9) we have

$$H_0(\cdot,r)|_{\partial B_{r_0}(Q_i)\cap L_{\beta_i+\nu}}=\mathrm{id}.$$

Now let $X_1(u)$ be a vector field defined as

$$X_1(u)=(0,P^+,0)$$

and $G_1(u,r)$ solve the flow equation

$$\frac{\mathrm{d}}{\mathrm{d}r}G_1(u,r)=X_1(G_1(u,r)),\quad 0\le r\le\delta^{-1};\quad G_1(u,0)=u.$$

We set $\tilde{G}_1(u,r)=G_1(u,r)+c(G_1)\in C_*^\infty$. Notice that $X_1$ is transversal to $\partial B_{r_0}(Q_i)$ within $L_{\beta_i+\nu}$. Moreover, for any $u\in B_{r_0}(Q_i)\cap L_{\beta_i+\nu}$ with $|P^+|\ge\delta r_0$ there holds $\tilde{G}_1(u,\delta^{-1})\notin B_{r_0}(Q_i)$; hence there exists a first time $0\le r=r_1(u)\le\delta^{-1}$ such that $\tilde{G}_1(u,r)\notin B_{r_0}(Q_i)$, and the map $u\mapsto r_1(u)$ is continuous. We extend this map continuously to the whole set $B_{r_0}(Q_i)\cap L_{\beta_i+\nu}$ by letting $r_1(u)=\delta^{-1}$ whenever $\tilde{G}_1(u,r)\in B_{r_0}(Q_i)$ for all $r\in[0,\delta^{-1}]$. Let $H_1(u,r)=\tilde{G}_1(u,\min\{r,r_1(u)\})$ and $u_r=G_1(u,\min\{r,r_1(u)\})$, then with a uniform constant $C_6>0$, Lemma 8.6 *(c)* shows

$$\frac{\mathrm{d}}{\mathrm{d}r}E_f[H_1(u,r)]=\frac{\mathrm{d}}{\mathrm{d}r}E_f[u_r]=\partial_P E_f[u_r]\cdot P^+\le-2|P^+|^2f(Q_i)^{-1}+Cr_0^3\le-c_3r_0^2,$$

if $|P^+|>\delta r_0$. Hence we choose $H$ to be the composition of $H_0$ with $H_1$, and for sufficiently small $r_0>0$ obtain a homotopy $H:\overline{B_{r_0}(Q_i)}\cap L_{\beta_i+\nu}\times[0,1]\to\overline{B_{r_0}(Q_i)}\cap L_{\beta_i+\nu}$ such that

$$H(B_{r_0}(Q_i)\cap L_{\beta_i+\nu},1)\subset B^+_{\delta r_0}\cup(\partial B_{r_0}(Q_i)\cap L_{\beta_i+\nu})$$

and

$$H(\cdot,r)|_{\partial B_{r_0}(P_i)\cap L_{\beta_i+\nu}}=\mathrm{id},\ 0\le r\le 1.$$

We compose $H$ with $U(T,\cdot)$ where $T = T(u_0)$ is defined in (*ii.*). Again by the estimate (8.8) we know that the length of time $\tau$ needed for the flow $U(t,\cdot)$ to travel the annular region $L_{\beta_i+\nu} \cap B_{r_0}(Q_i) \setminus B_{r_0/2}(Q_i)$ is uniformly positive. Therefore, we conclude that for sufficiently small $\nu > 0$ we have $U(T, \partial B_{r_0}(Q_i) \cap L_{\beta_i+\nu}) \subset L_{\beta_i-\nu}$. The remaining proof can be proceeded as in Lemma 8.7. □

Finally, our Proposition 8.2 follows from Lemmas 8.3, 8.4, 8.7 and 8.8. All in all, we have finished the whole program.

## Appendix A. Short time existence

As a first step to this goal, for a given initial datum $u_0$ and any constant $\kappa > \|u_0\|_{H^2(\mathbb{S}^2)}$ we define a set, for any $T > 0$, as follows

$$H := \left\{ u(t,\cdot) \in L^\infty([0,T], H^2(\mathbb{S}^2)); u(0,\cdot) = u_0(\cdot) \ \text{ and } \ \|u\|_{L^\infty([0,T],H^2(\mathbb{S}^2))} < \kappa \right\}.$$

Now for any function $u \in H$, we also define two functions:

$$\alpha(t) = \left( \int_{\mathbb{S}^2} f^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \right)^{-1} \int_{\mathbb{S}^2} f(-\Delta_{\mathbb{S}^2} u + 1) \mathrm{d}\mu_{\mathbb{S}^2}$$

and

$$F = \alpha f - e^{-2u}.$$

**Lemma A.1.** $F \in L^\infty([0,T], H^2(\mathbb{S}^2))$.

**Proof.** As the first preparation, we check the boundedness of the function $\alpha(t)$ for any $0 < t < T$.

On one hand, a simple Hölder's inequality implies

$$\begin{aligned} & \left( \int_{\mathbb{S}^2} f(-\Delta_{\mathbb{S}^2} u + 1) \mathrm{d}\mu_{\mathbb{S}^2} \right)^2 \\ & \leq 2 \left( \int_{\mathbb{S}^2} f \Delta_{\mathbb{S}^2} u \mathrm{d}\mu_{\mathbb{S}^2} \right)^2 + 2 \left( \int_{\mathbb{S}^2} f \mathrm{d}\mu_{\mathbb{S}^2} \right)^2 \\ & \leq 2 \left( \int_{\mathbb{S}^2} f^2 \mathrm{d}\mu_{\mathbb{S}^2} \right) [\|u\|^2_{H^2(\mathbb{S}^2)} + 4\pi] \\ & \leq C, \end{aligned}$$

with $C$ depending only on $\|f\|_{L^2(\mathbb{S}^2)}$ and $\kappa$.

On the other hand, due to the dimension restriction, $u \in H$ forces the existence of a positive constant $S_0$ such that $\|u\|_{L^\infty(\mathbb{S}^2)} \leq S_0\kappa$. Thus we clearly have

$$e^{2u} \geq e^{-2\|u\|_{L^\infty(\mathbb{S}^2)}} \geq e^{-2S_0\kappa}$$

which makes us easily conclude that

$$\int\limits_{\mathbb{S}^2} f^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} \geq e^{-2S_0\kappa} \int\limits_{\mathbb{S}^2} f^2 \mathrm{d}\mu_{\mathbb{S}^2}.$$

The above two estimates, together with definition of $\alpha$, yield

$$|\alpha| \leq C := C(\|f\|_{L^2(\mathbb{S}^2)}, \kappa)$$

Now, with $f$ a given smooth function on $\mathbb{S}^2$, it is not hard to see that $F \in L^2(\mathbb{S}^2)$. With the same observation as above, namely, there exists a constant $S_0 > 0$ such that, if $u \in H$, then $\max_{[0,T]\times\mathbb{S}^2} |u| \leq S_0\kappa$.

Next, again due to the smoothness of $f$, $\alpha(t)\Delta_{\mathbb{S}^2} f \in L^2(\mathbb{S}^2)$. Notice that $\Delta_{\mathbb{S}^2}(e^{-2u}) = e^{-2u}[-2\Delta_{\mathbb{S}^2} u + 4|\nabla u|^2]$. Since $u \in H$, the first term is in $L^2(\mathbb{S}^2)$. Now standard Sobolev inequality, note that $n = 2$, implies

$$\begin{aligned}
&[\int\limits_{\mathbb{S}^2} (|\nabla u|^2_{\mathbb{S}^2})^2 \mathrm{d}\mu_{\mathbb{S}^2}]^{\frac{1}{2}} \leq K(n,1) \int\limits_{\mathbb{S}^2} |\nabla|\nabla u|^2|_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int\limits_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \\
\leq& 2K(n,1) \int\limits_{\mathbb{S}^2} |\nabla u|_{\mathbb{S}^2} |\nabla|\nabla u||_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int\limits_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \\
\leq& (2K(n,1) + 1)[\int\limits_{\mathbb{S}^2} |\nabla u|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}]^{\frac{1}{2}} \|u\|_{H^2(\mathbb{S}^2)}.
\end{aligned}$$

Clearly this implies the second term in $\Delta_{\mathbb{S}^2}(e^{-2u})$ is also in $L^2(\mathbb{S}^2)$ so is itself. Thus the proof of Lemma is complete. □

**Remark A.2.** In general, by [1, Theorem 2.14], we should have the following inequality which will be freely used without proof: if $\psi \in H^1(\mathbb{S}^n)$ with $\int_{\mathbb{S}^2} \psi \mathrm{d}\mu_{\mathbb{S}^2} = 0$ and $n \geq 2$, then

$$\left( \int\limits_{\mathbb{S}^n} \psi^{\frac{2n}{n-1}} \mathrm{d}\mu_{\mathbb{S}^n} \right)^{\frac{n-1}{n}} \leq 2K(n,1) \|\psi\|_{L^2(\mathbb{S}^n)} \|\nabla\psi\|_{L^2(\mathbb{S}^n)}, \tag{A.1}$$

where

$$K(n,1) = \frac{1}{n} \left( \frac{n}{\omega_{n-1}} \right)^{\frac{1}{n}}, \qquad \omega_{n-1} = \mathrm{Vol}(S^{n-1}).$$

Next for each $u \in H$, consider the Cauchy problem of the following parabolic equation:

$$\begin{cases} v_t - \Delta_g v = \alpha f - e^{-2u} := F, \\ v(0) = u_0, \end{cases} \tag{A.2}$$

where $g = e^{2u} g_{\mathbb{S}^2}$ and $\Delta_g = e^{-2u} \Delta_{\mathbb{S}^2}$.

**Lemma A.3.** *$v \in H$ if the time $T$ is chosen sufficiently small.*

**Proof.** As the first step to our goal, multiplying Eq. (A.2) by $-\Delta_g v$ and integrating it over $\mathbb{S}^2$ with measure $\mathrm{d}\mu_g$, we can display the identity:

$$-\int_{\mathbb{S}^2} v_t \Delta_{\mathbb{S}^2} v \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (\Delta_g v)^2 \mathrm{d}\mu_g = -\int_{\mathbb{S}^2} \Delta_g v F \mathrm{d}\mu_g.$$

Up to an integration by parts, together with Young's inequality, this identity implies the following inequality

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2} |\nabla v|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (\Delta_g v)^2 \mathrm{d}\mu_g \le \frac{1}{4}\int_{\mathbb{S}^2} F^2 \mathrm{d}\mu_g + \int_{\mathbb{S}^2} (\Delta_g v)^2 \mathrm{d}\mu_g.$$

Observe that $\int_{\mathbb{S}^2} F^2 \mathrm{d}\mu_g \le e^{2S_0\kappa} \int_{\mathbb{S}^2} F^2 \mathrm{d}\mu_{\mathbb{S}^2}$. Thus we conclude that

$$\int_{\mathbb{S}^2} |\nabla v(t)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}(t) \le \frac{e^{2S_0\kappa}}{2}\int_0^t \int_{\mathbb{S}^2} F^2 \mathrm{d}\mu_{\mathbb{S}^2} \mathrm{d}\tau + \int_{\mathbb{S}^2} |\nabla v(0)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}. \tag{A.3}$$

Now Lemma A.1 implies that

$$\frac{e^{2S_0\kappa}}{2}\int_0^t \int_{\mathbb{S}^2} F^2 \mathrm{d}\mu_{\mathbb{S}^2} \mathrm{d}\tau \le \left[\frac{e^{2S_0\kappa}}{2} \max_{0\le t\le T} \|F\|^2_{L^2(\mathbb{S}^2)}\right] t.$$

Secondly, multiply Eq. (A.2) by $v$ and integrate it over $\mathbb{S}^2$ with measure $\mathrm{d}\mu_{\mathbb{S}^2}$, with the help of the integration by parts, to display the identity

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2} v^2 \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} e^{-2u} |\nabla v|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} = \int_{\mathbb{S}^2} F v \mathrm{d}\mu_{\mathbb{S}^2} + 2\int_{\mathbb{S}^2} v e^{-2u} \langle \nabla u, \nabla v \rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}. \tag{A.4}$$

We estimate the first term on the right hand side of Eq. (A.4) through Hölder's inequality:

$$\int_{\mathbb{S}^2} F v \mathrm{d}\mu_{\mathbb{S}^2} \le \frac{1}{2}\int_{\mathbb{S}^2} F^2 \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{2}\int_{\mathbb{S}^2} v^2 \mathrm{d}\mu_{\mathbb{S}^2}$$

and use the inequality (A.1) and to bound the second term

$$
\begin{aligned}
&\left|-\frac{1}{2}\int_{\mathbb{S}^2}\langle\nabla e^{-2u},\nabla v^2\rangle_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\right|\\
=&\left|\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}e^{-2u})(v-\bar{v})^2\mathrm{d}\mu_{\mathbb{S}^2}+2\bar{v}\int_{\mathbb{S}^2}\langle e^{-2u},\nabla v\rangle\mathrm{d}\mu_{\mathbb{S}^2}\right|\\
\leq&\left(\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}e^{-2u})^2\mathrm{d}\mu_{\mathbb{S}^2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^2}(v-\bar{v})^4\mathrm{d}\mu_{\mathbb{S}^2}\right)^{\frac{1}{2}}\\
&+2|\bar{v}|\left(\int_{\mathbb{S}^2}|\nabla e^{-2u}|^2\mathrm{d}\mu_{\mathbb{S}^2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{S}^2}|\nabla v|^2\mathrm{d}\mu_{\mathbb{S}^2}\right)^{\frac{1}{2}}\\
\leq& e^{S_0\kappa}\left(2K(2,1)\|\Delta_{\mathbb{S}^2}e^{-2u}\|_{L^2(\mathbb{S}^2)}+2\|\nabla e^{-2u}\|_{L^2(\mathbb{S}^2)}\right)\|v\|_{L^2(\mathbb{S}^2)}\|e^{-u}\nabla v\|_{L^2(\mathbb{S}^2)},
\end{aligned}
$$

where $\bar{v}=\fint_{\mathbb{S}^2}v\mathrm{d}\mu_{\mathbb{S}^2}$. Here in the last step, we have used the boundedness of $\|u\|_\infty$ and the $W^{1,1}$ version of Sobolev embedding, i.e. (A.1).

Now as we have shown in the proof of Lemma A.1, we have

$$
2K(2,1)\|\Delta_{\mathbb{S}^2}e^{-2u}\|_{L^2(\mathbb{S}^2)}+2\|\nabla e^{-2u}\|_{L^2(\mathbb{S}^2)}\leq C
$$

for some constant $C$ depending only on $\kappa$.

Thus, by Eq. (A.4) we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2}v^2\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}e^{-2u}|\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\\
\leq&\frac{1}{2}\int_{\mathbb{S}^2}F^2\mathrm{d}\mu_{\mathbb{S}^2}+\frac{1}{2}\int_{\mathbb{S}^2}v^2\mathrm{d}\mu_{\mathbb{S}^2}+C\|v\|_{L^2(\mathbb{S}^2)}\left\|e^{-u}\nabla v\right\|_{L^2(\mathbb{S}^2)}\\
\leq&\frac{1}{2}\int_{\mathbb{S}^2}F^2\mathrm{d}\mu_{\mathbb{S}^2}+\frac{C^2+1}{2}\int_{\mathbb{S}^2}v^2\mathrm{d}\mu_{\mathbb{S}^2}+\frac{1}{2}\int_{\mathbb{S}^2}e^{-2u}|\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2},
\end{aligned}
$$

which implies, with some positive constant $\kappa_1=1+C^2$, that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2}v^2\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}e^{-2u}|\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\leq\int_{\mathbb{S}^2}F^2\mathrm{d}\mu_{\mathbb{S}^2}+\kappa_1\int_{\mathbb{S}^2}v^2\mathrm{d}\mu_{\mathbb{S}^2}.
$$

Then the above inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\kappa_1 t}\int\limits_{\mathbb{S}^2} v^2 \mathrm{d}\mu_{\mathbb{S}^2}\right) + e^{-\kappa_1}\int\limits_{\mathbb{S}^2} e^{-2u}|\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} \le \int\limits_{\mathbb{S}^2} F^2 \mathrm{d}\mu_{\mathbb{S}^2} e^{-\kappa_1 t}.$$

Integrating the above inequality over $(0, t)$ to show

$$\begin{aligned}&\int\limits_{\mathbb{S}^2} v(t)^2 \mathrm{d}\mu_{\mathbb{S}^2} + \int\limits_0^t e^{\kappa_1(t-\tau)}\int\limits_{\mathbb{S}^2} e^{-2u}|\nabla v|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}\tau \\ \le& e^{\kappa_1 t}\int\limits_0^t\int\limits_{\mathbb{S}^2} F^2 e^{-\kappa_1\tau}\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}\tau + e^{\kappa_1 t}\int\limits_{\mathbb{S}^2} v(0)^2\mathrm{d}\mu_{\mathbb{S}^2}. \end{aligned} \tag{A.5}$$

Finally we set $w = \Delta_{\mathbb{S}^2} v$. Then a direct computation shows

$$w_t - \Delta_g w = \Delta_{\mathbb{S}^2} F + (\Delta_{\mathbb{S}^2} e^{-2u})w + 2\nabla e^{-2u}\cdot\nabla w. \tag{A.6}$$

Notice that, by Lemma A.1, $\Delta_{\mathbb{S}^2} F \in L^2(\mathbb{S}^2)$. Therefore Eq. (A.6) is similar to Eq. (A.2) except for the extra term $(\Delta_{\mathbb{S}^2} e^{-2u})w + 2\langle\nabla e^{-2u}, \nabla w\rangle_{\mathbb{S}^2}$. This term looks unfavorable to us since it might not be in $L^2(\mathbb{S}^2)$. Since we only need to get the $L^2$ norm bound on $w$, as was concerned on this issue, this extra term indeed does not cost any trouble for us. Namely we simply observe that

$$\int\limits_{\mathbb{S}^2}[(\Delta_{\mathbb{S}^2} e^{-2u})w + 2\nabla e^{-2u}\cdot\nabla w]w d\mu_{\mathbb{S}^2} = 0.$$

Thus we have no further difficulty to repeat the above $L^2(\mathbb{S}^2)$ estimate for $v$ to get the similar $L^2(\mathbb{S}^2)$ estimate for $w$:

$$\begin{aligned}&\int\limits_{\mathbb{S}^2} w(t)^2 \mathrm{d}\mu_{\mathbb{S}^2} + \int\limits_0^t e^{\kappa_1(t-\tau)}\int\limits_{\mathbb{S}^2} e^{-2u}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}\tau \\ \le& e^{\kappa_1 t}\int\limits_0^t\int\limits_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2}F)^2 e^{-\kappa_1\tau}\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}\tau + e^{\kappa_1 t}\int\limits_{\mathbb{S}^2} w(0)^2\mathrm{d}\mu_{\mathbb{S}^2}. \end{aligned} \tag{A.7}$$

At last, combine Eqs. (A.3), (A.5) and (A.7) to obtain

$$\max_{0\le t\le T}\|v\|_{H^2(\mathbb{S}^2)} \le e^{\kappa_1 T} T \max_{0\le t\le T}\|F\|_{H^2(\mathbb{S}^2)} + e^{\kappa_1 T}\|v(0)\|_{H^2(\mathbb{S}^2)}.$$

This shows that we can select $T$ sufficiently small such that $\max_{0\le t\le T}\|v\|_{H^2(\mathbb{S}^2)} < \kappa$. Notice that $v(0) = u_0$, hence $v \in H$. The proof of Lemma A.3 is complete. □

As a byproduct of Eq. (A.6), we also obtain the following useful estimate.

**Lemma A.4.** *If $u_0 \in H^3(\mathbb{S}^2)$, then there exists $\kappa_2 > 0$, for all $0 \le t \le T$,*

$$\int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \le 2e^{s_0\kappa} e^{\kappa_2 t} \int_0^t \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} F)^2 e^{-\kappa_2 \tau} \mathrm{d}\mu_{\mathbb{S}^2} \mathrm{d}\tau + e^{\kappa_2 t} \int_{\mathbb{S}^2} |\nabla w(0)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}, \tag{A.8}$$

*where $w = \Delta_{\mathbb{S}^2} v$.*

**Proof.** Multiply (A.6) by $(-\Delta_{\mathbb{S}^2} w)$ to obtain, through an integration by parts,

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}^2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} e^{-2u} (\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
=& \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} F) \cdot (-\Delta_{\mathbb{S}^2} w) \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u}) w (-\Delta_{\mathbb{S}^2} w) \mathrm{d}\mu_{\mathbb{S}^2} \\
&- 2\int_{\mathbb{S}^2} \langle \nabla e^{-2u}, \nabla w \rangle_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} w) \mathrm{d}\mu_{\mathbb{S}^2} \\
:=& I + II + III.
\end{aligned} \tag{A.9}$$

Of course, our aim is to control all three terms by controlled information. For the first term, Hölder's inequality provides

$$|I| \le 4 \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} F)^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{4} \int_{\mathbb{S}^2} e^{-2u} (\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2}.$$

The fact that $⨍_{\mathbb{S}^2} w \mathrm{d}\mu_{\mathbb{S}^2} = 0$, together with Green's representation, gives the $L^\infty(\mathbb{S}^2)$ control on $w$

$$\|w\|_{L^\infty(\mathbb{S}^2)} \le \rho_0 \left( ⨍_{\mathbb{S}^2} |\nabla w|^4_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \right)^{\frac{1}{4}},$$

where $\rho_0 = (⨍_{\mathbb{S}^2} |\nabla G|^{4/3}_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^{3/4}$ with $G$ the Green's function for $-\Delta_{\mathbb{S}^2}$. The well-known fact is that $\rho_0 < +\infty$. Therefore, the second term can be estimated as follows

$$\begin{aligned}
|II| &\le 4 \int_{\mathbb{S}^2} e^{2u} (\Delta_{\mathbb{S}^2} e^{-2u})^2 w^2 \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{4} \int_{\mathbb{S}^2} e^{-2u} (\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
&\le 4\rho_0^2 \int_{\mathbb{S}^2} e^{2u} (\Delta_{\mathbb{S}^2} e^{-2u})^2 \mathrm{d}\mu_{\mathbb{S}^2} (⨍_{\mathbb{S}^2} |\nabla w|^4_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + \frac{1}{4} \int_{\mathbb{S}^2} e^{-2u} (\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned}$$

While for $III$, the routine estimate goes to provide

$$|III| \le 4\int_{\mathbb{S}^2} \langle \nabla e^{-2u}, \nabla w\rangle_{\mathbb{S}^2}^2 e^{2u} \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{4}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2}$$
$$\le 4(\int_{\mathbb{S}^2} |\nabla e^{-2u}|_{\mathbb{S}^2}^4 e^{4u} \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} (\int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^4 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + \frac{1}{4}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2}$$

For simple notation, set $\beta_0 = 4\rho_0^2 \int_{\mathbb{S}^2} e^{2u}(\Delta_{\mathbb{S}^2} e^{-2u})^2 \mathrm{d}\mu_{\mathbb{S}^2} + 4(\int_{\mathbb{S}^2} |\nabla e^{-2u}|_{\mathbb{S}^2}^4 e^{4u} \mathrm{d}\mu_{\mathbb{S}^2})^{1/2}$ to obtain

$$\begin{aligned}
&|II| + |III| \\
\le&\beta_0(\int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^4 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + \frac{1}{2}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
\le&\beta_0(\int_{\mathbb{S}^2} (|\nabla w|_{\mathbb{S}^2} - \overline{|\nabla w|_{\mathbb{S}^2}})^4 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + 3\beta_0 \|\nabla w\|_{L^2(\mathbb{S}^2)}^2 + \frac{1}{2}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
\le&2\beta_0 K(2,1)\|\nabla^2 w\|_{L^2(\mathbb{S}^2)} \|\nabla w\|_{L^2(\mathbb{S}^2)} + 3\beta_0 \|\nabla w\|_{L^2(\mathbb{S}^2)}^2 + \frac{1}{2}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
\le&2\beta_0 K(2,1)(\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} (\int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^2 \mathrm{d}\mu_{\mathbb{S}^2})^{\frac{1}{2}} + 3\beta_0 \int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
&+ \frac{1}{2}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
\le&(16K^2(2,1)e^{s_0\kappa}\beta_0^2 + 3\beta_0)\int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^2 \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{4}\int_{\mathbb{S}^2} e^{-2s_0\kappa}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
&+ \frac{1}{2}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
\le&\frac{\kappa_2}{2}\int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^2 \mathrm{d}\mu_{\mathbb{S}^2} + \frac{3}{4}\int_{\mathbb{S}^2} e^{-2u}(\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2},
\end{aligned}$$

where $\overline{|\nabla w|_{\mathbb{S}^2}} = ⨍_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}$ and $\frac{\kappa_2}{2} = 16K^2(2,1)e^{s_0\kappa}\beta_0^2 + 3\beta_0$ and also we observe the fact that, for standard metric on $\mathbb{S}^2$,

$$\int_{\mathbb{S}^2} |\nabla^2 w|_{\mathbb{S}^2}^2 \mathrm{d}\mu_{\mathbb{S}^2} = \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} w)^2 \mathrm{d}\mu_{\mathbb{S}^2} - \int_{\mathbb{S}^2} |\nabla w|_{\mathbb{S}^2}^2 \mathrm{d}\mu_{\mathbb{S}^2}$$

by Bochner identity. Then put $(|I| + |II| + |III|)$ into Eq. (A.9) to show

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}e^{-2u}(\Delta_{\mathbb{S}^2}w)^2\mathrm{d}\mu_{\mathbb{S}^2}$$
$$\leq\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}F)^2e^{2u}\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}e^{-2u}(\Delta_{\mathbb{S}^2}w)^2\mathrm{d}\mu_{\mathbb{S}^2}+\frac{\kappa_2}{2}\int_{\mathbb{S}^2}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}.$$

Cancel the term $\int_{\mathbb{S}^2}e^{-2u}(\Delta_{\mathbb{S}^2}w)^2\mathrm{d}\mu_{\mathbb{S}^2}$ on both sides and multiply the number 2 to have:

$$\frac{\mathrm{d}}{\mathrm{d}t}(e^{-\kappa_2 t}\int_{\mathbb{S}^2}|\nabla w|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2})\leq 2\int_{\mathbb{S}^2}(\Delta_{\mathbb{S}^2}F)^2\mathrm{d}\mu_{\mathbb{S}^2}e^{2s_0\kappa}e^{-\kappa_2 t}.$$

Upon the integration, one concludes the inequality (A.8). □

From the above discussion, we define a map $v:=\Psi(u)$ for each $u\in H$. Our purpose is to show that $\Psi$ is a contraction map on $H$.

**Lemma A.5.** *The map $\Psi$ defined above is a contraction with contraction coefficient $1/2$ if $T$ is suitably small.*

**Proof.** For $u_1,u_2\in H$, let $v_i=\Psi(u_i)$ for $1\leq i\leq 2$, then

$$(v_i)_t-\Delta_{g_i}v_i=F_i \quad \text{and} \quad v_i(0)=u_0,$$

where $g_i=e^{2u_i}g_{\mathbb{S}^2}$, $\mathrm{d}\mu_i=\mathrm{d}\mu_{g_i}$, and

$$F_i=\alpha_i f-e^{-2u_i},\quad \alpha_i=\frac{\int_{\mathbb{S}^2}f(-\Delta_{\mathbb{S}^2}u_i+1)\mathrm{d}\mu_{\mathbb{S}^2}}{\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_i}\quad \text{for } i=1,2.$$

Thus, we obtain the parabolic equation:

$$\begin{cases}(v_2-v_1)_t-\Delta_{g_2}v_2+\Delta_{\mathbb{S}^2}v_1=(\alpha_2-\alpha_1)f+e^{-2u_1}-e^{-2u_2}:=\tilde{F},\\(v_2-v_1)(0)=0.\end{cases}\tag{A.10}$$

**Step 1.** We show that $\max_{0\leq t\leq T}\|\tilde{F}\|_{H^2(\mathbb{S}^2)}\leq C\|u_2-u_1\|_{H^2(\mathbb{S}^2)}$.
As a first approach, due to $u_i,v_i\in H$ for $i=1,2$, we have

$$\alpha_2-\alpha_1=\frac{\int_{\mathbb{S}^2}f^2d\mu_1\int_{\mathbb{S}^2}f(-\Delta_{\mathbb{S}^2}u_2+1)\mathrm{d}\mu_{\mathbb{S}^2}-\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_2\int_{\mathbb{S}^2}f(-\Delta_{\mathbb{S}^2}u_1+1)\mathrm{d}\mu_{\mathbb{S}^2}}{\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_1\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_2}$$

$$=\frac{\left(\int_{\mathbb{S}^2}f\mathrm{d}\mu_{\mathbb{S}^2}-\int_{\mathbb{S}^2}f\Delta_{\mathbb{S}^2}u_1\mathrm{d}\mu_{\mathbb{S}^2}\right)\int_{\mathbb{S}^2}f^2(e^{2u_1}-e^{2u_2})\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_1\int_{\mathbb{S}^2}f\Delta_{\mathbb{S}^2}(u_1-u_2)\mathrm{d}\mu_{\mathbb{S}^2}}{\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_1\int_{\mathbb{S}^2}f^2\mathrm{d}\mu_2}$$

and

$$\left|\int\limits_{\mathbb{S}^2} f^2(e^{2u_1}-e^{2u_2})\mathrm{d}\mu_{\mathbb{S}^2}\right| = \left|\int\limits_{\mathbb{S}^2} f^2 e^{2u_2}(e^{2(u_1-u_2)}-1)\mathrm{d}\mu_{\mathbb{S}^2}\right|$$
$$\leq C\,\|u_1-u_2\|_{L^\infty(\mathbb{S}^2)} \leq C\,\|u_1-u_2\|_{H^2(\mathbb{S}^2)}\,.$$

Then

$$(\alpha_1-\alpha_2)^2 \leq C\,\|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}\,.$$

Similarly,

$$\int\limits_{\mathbb{S}^2}(e^{2(u_1-u_2)}-1)^2(\Delta_{g_1}v_1)^2\mathrm{d}\mu_{\mathbb{S}^2} \leq C\,\|u_1-u_2\|^2_{L^\infty(\mathbb{S}^2)}\,\|v_1\|^2_{H^2(\mathbb{S}^2)} \leq C\,\|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}\,,$$
$$\int\limits_{\mathbb{S}^2}|e^{-2u_1}-e^{-2u_2}|^2\mathrm{d}\mu_{\mathbb{S}^2} \leq C\,\|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}\,.$$

Thus, we conclude that

$$\int\limits_{\mathbb{S}^2}\tilde{F}^2\mathrm{d}\mu_{\mathbb{S}^2} \leq C\,\|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}\,,$$

with $C$ depending on $f$ and $\kappa$.

Now with direct calculation, it should have no difficulty seeing that

$$\max_{0\leq t\leq T}\|\tilde{F}\|^2_{H^2(\mathbb{S}^2)} \leq C\,\|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}\,. \tag{A.11}$$

Eq. For simplicity, we set $h=v_2-v_1$. Basically the rest of the argument here is just to repeat the proof of Lemma A.3. We do it in several steps:

**Step 2.** As before, now multiply Eq. (A.10) by $-\Delta_{\mathbb{S}^2}h$ and integrate it over $\mathbb{S}^2$ with measure $\mathrm{d}\mu_{\mathbb{S}^2}$ to show

$$-\int\limits_{\mathbb{S}^2}h_t\Delta_{\mathbb{S}^2}h\mathrm{d}\mu_{\mathbb{S}^2}+\int\limits_{\mathbb{S}^2}(\Delta_{g_2}h)^2\mathrm{d}\mu_{g_2}+\int\limits_{\mathbb{S}^2}(e^{-2u_2}-e^{-2u_1})(\Delta_{\mathbb{S}^2}v_1)\Delta_{\mathbb{S}^2}h\mathrm{d}\mu_{\mathbb{S}^2}$$
$$=-\int\limits_{\mathbb{S}^2}\Delta_{g_2}h\tilde{F}\mathrm{d}\mu_{g_2}.$$

An integration by parts, together with Young's inequality, shows

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_{\mathbb{S}^2}|\nabla h|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}+\int\limits_{\mathbb{S}^2}(\Delta_{g_2}h)^2\mathrm{d}\mu_{g_2} \leq \frac{1}{2}\int\limits_{\mathbb{S}^2}\tilde{F}^2\mathrm{d}\mu_{g_2}+\int\limits_{\mathbb{S}^2}(\Delta_{g_2}h)^2\mathrm{d}\mu_{g_2}+C\,\|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}\,,$$

with $C$ depending only on $\kappa$.

Thus we conclude that, with the help of $h(0)=0$ and (A.11),

$$\int_{\mathbb{S}^2} |\nabla h(t)|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}(t) \le CT \|u_1-u_2\|^2_{H^2(\mathbb{S}^2)}.$$

**Step 3.** Next, multiply Eq. (A.10) by $h$ and then integrate it over $\mathbb{S}^2$ with measure $\mathrm{d}\mu_{\mathbb{S}^2}$ to display the identity

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2} h^2 \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} e^{-2u_2}|\nabla h|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2} \\
=&\int_{\mathbb{S}^2} \tilde{F} h \mathrm{d}\mu_{\mathbb{S}^2} + 2\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2}) h^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
&+\int_{\mathbb{S}^2} (e^{-2u_2}-e^{-2u_1})(\Delta_{\mathbb{S}^2} v_1) h \mathrm{d}\mu_{\mathbb{S}^2}.
\end{aligned} \tag{A.12}$$

We estimate the first term on the right hand side of Eq. (A.12) through Hölder's inequality:

$$\left|\int_{\mathbb{S}^2} \tilde{F} h \mathrm{d}\mu_{\mathbb{S}^2}\right| \le \frac{1}{2}\int_{\mathbb{S}^2} \tilde{F}^2 \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{2}\int_{\mathbb{S}^2} h^2 \mathrm{d}\mu_{\mathbb{S}^2}$$

and use the well-known inequality (A.1) to bound the second term

$$\begin{aligned}
&\left|\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2}) h^2 \mathrm{d}\mu_{\mathbb{S}^2}\right| \\
=&\left|\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2}) (h-\bar{h})^2 \mathrm{d}\mu_{\mathbb{S}^2} + 2\bar{h}\int_{\mathbb{S}^2} \langle \nabla e^{-2u_2}, \nabla h\rangle \mathrm{d}\mu_{\mathbb{S}^2}\right| \\
\le&\left(\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2})^2 \mathrm{d}\mu_{\mathbb{S}^2}\right)^{\frac{1}{2}} \left(\int_{\mathbb{S}^2} (h-\bar{h})^4 \mathrm{d}\mu_{\mathbb{S}^2}\right)^{\frac{1}{2}} + 2\|h\|_{L^2(\mathbb{S}^2)}\|\nabla e^{-2u_2}\|_{L^2(\mathbb{S}^2)}\|\nabla h\|_{L^2(\mathbb{S}^2)} \\
\le& C\|h\|_{L^2(\mathbb{S}^2)}\|e^{-u}\nabla h\|_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

where $\bar{h} = ⨍_{\mathbb{S}^2} h \mathrm{d}\mu_{\mathbb{S}^2}$ and we have used the $W^{1,1}$ version of Sobolev embedding, i.e. (A.1), and the estimate from Lemma A.1,

$$2K(2,1)e^{S_0\kappa}\|\Delta_{\mathbb{S}^2} e^{-2u_2}\|_{L^2(\mathbb{S}^2)} + 2e^{S_0\kappa}\|\nabla e^{-2u_2}\|_{L^2(\mathbb{S}^2)} \le C$$

for some constant $C$ depending only on $\kappa$.

Comparing with previous case just for $v$, here we have an extra term need to be settled: the last term in equation (A.12). However it is not hard to check that, in fact, we have

$$\left|\int_{\mathbb{S}^2}(e^{-2u_2}-e^{-2u_1})(\Delta_{\mathbb{S}^2}v_1)h\mathrm{d}\mu_{\mathbb{S}^2}\right|\le C_1\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}+\frac{1}{2}\|h\|^2_{L^2(\mathbb{S}^2)},$$

by Hölder's and Young's inequalities, together with the observation that $\Delta_{\mathbb{S}^2}v_1\in L^2(\mathbb{S}^2)$.

Thus, by Eq. (A.12) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}e^{-2u_2}|\nabla h|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2}\\ \le&\frac{1}{2}\int_{\mathbb{S}^2}\tilde{F}^2\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}+C\,\|h\|_{L^2(\mathbb{S}^2)}\left\|e^{-u_2}\nabla h\right\|_{L^2(\mathbb{S}^2)}\\ &+C\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}+C_1\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}\\ \le&\frac{C_2}{2}\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}+\frac{\kappa_3}{2}\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}+\int_{\mathbb{S}^2}e^{-2u_2}|\nabla h|^2_{\mathbb{S}^2}\mathrm{d}\mu_{\mathbb{S}^2},\end{aligned}$$

with $\kappa_3=2(1+C)^2$ and $C_2=2(C_1+C)$, this last constant $C$ arising from the estimate on $\tilde{F}$.

Thus the following estimate surely holds true

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}\le C_2\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}+\kappa_3\int_{\mathbb{S}^2}v^2\mathrm{d}\mu_{\mathbb{S}^2}.$$

Then the above inequality becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(e^{-\kappa_3t}\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}\right)\le C_2\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}e^{-\kappa_3t}.$$

Integrate the above inequality over $(0,t)$ to show

$$\int_{\mathbb{S}^2}h^2\mathrm{d}\mu_{\mathbb{S}^2}\le C_2e^{\kappa_3t}\int_0^t\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}e^{-\kappa_3\tau}\mathrm{d}\mu_{\mathbb{S}^2}\mathrm{d}\tau.$$

To get the expected estimate, we only need to observe that $\kappa_3>0$ and $\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}(\tau)\le\max_{0\le t\le T}\|u_2-u_1\|^2_{H^2(\mathbb{S}^2)}$. Therefore, we have

$$\int_{\mathbb{S}^2} h^2 \mathrm{d}\mu_{\mathbb{S}^2} \leq C_2 e^{\kappa_3 T} T \max_{0\leq t\leq T} \|u_2 - u_1\|^2_{H^2(\mathbb{S}^2)}.$$

**Step 4.** Finally we set $w = \Delta_{\mathbb{S}^2} h$. Then a direct computation shows:

$$w_t - \Delta_{\mathbb{S}^2}(e^{-2u_2}\Delta_{\mathbb{S}^2} v_2 - e^{-2u_1}\Delta_{\mathbb{S}^2} v_1) = \Delta_{\mathbb{S}^2}\tilde{F}.$$

From this, it follows that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
=& -\int_{\mathbb{S}^2} \langle \nabla w, \nabla[e^{-2u_2} w + (e^{-2u_2} - e^{-2u_1})(\Delta_{\mathbb{S}^2} v_1)]\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2}\tilde{F}) w \mathrm{d}\mu_{\mathbb{S}^2} \\
=& -\int_{\mathbb{S}^2} e^{-2u_2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + \frac{1}{2}\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2}) w^2 \mathrm{d}\mu_{\mathbb{S}^2} \\
&+ \int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2}\tilde{F}) w \mathrm{d}\mu_{\mathbb{S}^2} - \int_{\mathbb{S}^2} \langle \nabla w, \nabla[(e^{-2u_2} - e^{-2u_1})(\Delta_{\mathbb{S}^2} v_1)]\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \\
:=& I + II + III + IV.
\end{aligned}$$

The term $I$ is favorable which will be used to settle two bad terms later. The trick for term $II$ is the same as before

$$\begin{aligned}
\left|\frac{1}{2}\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2}) w^2 \mathrm{d}\mu_{\mathbb{S}^2}\right| &= \left|\frac{1}{2}\int_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} e^{-2u_2}) w^2 \mathrm{d}\mu_{\mathbb{S}^2}\right| \\
&\leq C\|w\|_{L^2(\mathbb{S}^2)} \|\nabla w\|_{L^2(\mathbb{S}^2)} + C\|w\|^2_{L^2(\mathbb{S}^2)}.
\end{aligned}$$

Third term can be treated just using Hölder's and Young's inequalities, together with the estimate (A.11).

A little bit difficult one is the term $IV$. First we use the integration by parts to transfer it to the now form as follows

$$\begin{aligned}
&\left|\int_{\mathbb{S}^2} \langle \nabla w, \nabla[(e^{-2u_2} - e^{-2u_1})(\Delta_{\mathbb{S}^2} v_1)]\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}\right| \\
\leq& \left|\int_{\mathbb{S}^2} (e^{-2u_2} - e^{-2u_1}) \langle \nabla w, \nabla \Delta v_1\rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2}\right| + \left|\int_{\mathbb{S}^2} \langle \nabla w, \nabla(e^{-2u_2} - e^{-2u_1})\rangle_{\mathbb{S}^2} (\Delta_{\mathbb{S}^2} v_1) \mathrm{d}\mu_{\mathbb{S}^2}\right| \\
:=& IV_1 + IV_2.
\end{aligned}$$

To treat term $IV_1$, we simply use Hölder's and Young's inequalities, together with (A.8), to get

$$\begin{aligned}
IV_1 :=& \left| \int_{\mathbb{S}^2} (e^{-2u_2} - e^{-2u_1}) \langle \nabla w, \nabla \Delta v_1 \rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \right| \\
\leq& \left| \int_{\mathbb{S}^2} (1 - e^{2(u_2-u_1)}) \langle e^{-u_2} \nabla w, e^{-u_2} \nabla \Delta v_1 \rangle_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \right| \\
\leq& \left( \int_{\mathbb{S}^2} e^{-2u_2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \right)^{\frac{1}{2}} \left[ \int_{\mathbb{S}^2} (1 - e^{2(u_2-u_1)})^2 e^{-2u_2} |\nabla \Delta v_1|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} \right]^{\frac{1}{2}} \\
\leq& \frac{1}{4} \int_{\mathbb{S}^2} e^{-2u_2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + C_3 \|u_2 - u_1\|^2_{H^2(\mathbb{S}^2)}.
\end{aligned}$$

For the last term $IV_2$, with the similar reasons, we can proceed the estimate as follows

$$\begin{aligned}
IV_2 :=& \left| \int_{\mathbb{S}^2} e^{-u_2} \langle \nabla w, \nabla (1 - e^{2(u_2-u_1)} e^{-2u_2}) \rangle_{\mathbb{S}^2} e^{u_2} (\Delta_{\mathbb{S}^2} v_1) \mathrm{d}\mu_{\mathbb{S}^2} \right| \\
\leq& \frac{1}{4} \int_{\mathbb{S}^2} e^{-2u_2} |\nabla w|^2_{\mathbb{S}^2} \mathrm{d}\mu_{\mathbb{S}^2} + C_4 \|u_2 - u_1\|^2_{H^2(\mathbb{S}^2)}.
\end{aligned}$$

Put all those estimates for terms $I$ to $IV$ together to conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2} \leq C_5 \|u_2 - u_1\|^2_{H^2(\mathbb{S}^2)} + \kappa_4 \int_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2},$$

with suitable positive constants $C_5$ and $\kappa_4$. Details will be left to readers to check.

Recall that $w(0) = 0$, hence integrate it to get

$$\max_{0 \leq t \leq T} \int_{\mathbb{S}^2} w^2 \mathrm{d}\mu_{\mathbb{S}^2} \leq C_5 e^{\kappa_4 T} T \max_{0 \leq t \leq T} \|u_2 - u_1\|^2_{H^2(\mathbb{S}^2)}.$$

Combine above all steps to conclude that, with suitable positive constants $\kappa_5$ and $C_6$,

$$\max_{0 \leq t \leq T} \|h\|_{H^2(\mathbb{S}^2)} \leq C_6 e^{\kappa_5 T} T \max_{0 \leq t \leq T} \|u_2 - u_1\|^2_{H^2(\mathbb{S}^2)}.$$

This completes the proof. □

**Theorem A.6.** *The flow equation $u_t = \alpha f - K$ defined as (2.2) is solvable on $[0, T]$ where $T$ is sufficiently small depending on the choice of initial data $u_0$ and $f$.*

**Proof.** Finally, we select $T$ small such that $C_6 e^{\kappa_5 T} T \leq 1/2$ to conclude that

$$\max_{0 \leq t \leq T} \|\Psi(u_1) - \Psi(u_2)\|_{L^\infty([0,T],H^2(\mathbb{S}^2))} \leq \frac{1}{2} \max_{0 \leq t \leq T} \|u_1 - u_2\|_{L^\infty([0,T],H^2(\mathbb{S}^2))}.$$

So, $\Psi$ is a contraction map within $H$. The contraction map theorem shows that there exists a unique $u \in H$ such that $\Psi(u) = u$. This is enough to claim above theorem. $\square$

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