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A new invariant on 3-dimensional manifolds and applications

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In this work, we study a functional involving the generalized scalar curvatures and prove a spherical sphere theorem under some pinching condition of this quantity. As an application, we define a new invariant on 3-dimensional manifolds and use it to study the topology of manifolds.

Keywords: Sphere theorem; fully nonlinear equation; space form.

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1. Introduction

The σ_k-scalar curvature, which was first introduced by Viaclovsky [13], has been intensively studied by many mathematicians. One of important applications is about the sphere theorem under an integral pinching condition. On 4-dimensional closed manifolds, such result is obtained by Chang–Gursky–Yang [3] and later on by Gursky–Viaclovsky [9]. A similar result on 3-dimensional closed manifolds is

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proved by Catino–Djadli [2] and Ge–Lin–Wang [4] with the different approach (see also [10, 1]).

Let us recall some definitions. Given an n-dimensional Riemannian manifold (M^n, g) ($n \geq 3$), let us denote by Ric (respectively R) the Ricci curvature (respectively the scalar curvature). The Schouten tensor of g is defined as

$$S_g := \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)} \cdot g\right).$$

For an integer k with $1 \leq k \leq n$, let σ_k be the kth elementary symmetric function in $\mathbb{R}^n$. The k-scalar curvature is defined by

$$\sigma_k(g) := \sigma_k(\Lambda_g),$$

where Λ_g is the set of eigenvalues of the matrix $g^{-1} \cdot S_g$. In particular, σ_1 is the classical scalar curvature up to a constant and σ_k for general k are generalized scalar curvatures, for example,

$$\sigma_1(g) = \frac{R}{2(n-1)}, \quad \sigma_2(g) = \frac{1}{2(n-2)^2}\left(-|\mathrm{Ric}|^2 + \frac{n}{4(n-1)}R^2\right).$$

To study almost Schur Lemma on 3-dimensional and 4-dimensional manifolds, in [5, 7], Ge and Wang study a new energy functional involving the generalized scalar curvatures $\sigma_1(g)$ and $\sigma_2(g)$ defined by

$$\mathcal{A}(M, g) := \frac{\mathrm{vol}(g) \cdot \displaystyle\int_M \sigma_2(g) dv_g}{\left(\displaystyle\int_M \sigma_1(g) dv_g\right)^2},$$

where $\mathrm{vol}(g)$ is the volume of the metric g. They showed that in the set of metrics with positive scalar curvatures, $\mathcal{A}(M^3, g)$ is bounded from above and there holds $\mathcal{A}(M^3, g) \leq \frac{1}{3}$ and the equality holds if and only if (M^3, g) is a space form with the positive scalar curvature. On the other hand, under the assumption $\mathcal{A}(M^3, g) \geq 0$, by the results in [2] and [4], there exists a metric $\widetilde{g}$ conformal to g such that $\mathrm{Ric}_{\widetilde{g}}$ is everywhere positive. Then using a result of Hamilton [12], it follows that M is diffeomorphic to a spherical space form.

With the help of the functional $\mathcal{A}(M, g)$, we try to understand the topology of M. Our main result is a sphere theorem as follows.

Theorem 1.1. *Let (M, g) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. Suppose that*

$$\mathcal{A}(M, g) > -\frac{23}{384}.$$

Then M is diffeomorphic to a spherical space form.

In the following, we improve this sphere theorem under more assumptions. In particular, when the scalar curvature is a positive constant, we could improve our sphere theorem as follows.

Theorem 1.2. *Let (M, g) be a closed 3-dimensional Riemannian manifold with constant positive scalar curvature. Suppose that*

$$\mathcal{A}(M, g) > -\frac{1}{3}.$$

Then M is diffeomorphic to a spherical space form.

We consider a product metric of $\mathbb{S}^1 \times \mathbb{S}^2$ (see Chap. 4, Sec. 4.2.2 in [11]). We could check that $\mathcal{A}(M, g) = -1$. It is well-known that $\mathbb{S}^1 \times \mathbb{S}^2$ is not diffeomorphic to a spherical space form.

Under a stronger assumption that g is the Yamabe metric, we will get the sharp lower bound of $\mathcal{A}(M, g)$ for which M is diffeomorphic to a spherical space form.

Theorem 1.3. *Let (M, g) be a closed 3-dimensional Riemannian manifold such that g is the Yamabe metric with constant positive scalar curvature. Suppose that*

$$\mathcal{A}(M, g) > -1.$$

Then M is diffeomorphic to a spherical space form.

Actually, Theorems 1.1 and 1.2 are both consequences of the following theorem.

Theorem 1.4. *Let (M, g) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. Suppose that*

$$\int_M \sigma_2(g) dv_g + \frac{1}{16 \operatorname{vol}(g)} \left(1 - \frac{(19 - 3\kappa)^2}{384}\right) \left(\int_M R dv_g\right)^2 > 0, \tag{1.1}$$

where $\kappa \in (0, 1]$ satisfying $\kappa \leq \frac{\operatorname{vol}(g) \min_M R}{\int_M R dv_g}$. Then there exists a conformal metric $g' \in [g]$ with positive Ricci curvature, in particular, M is diffeomorphic to a spherical space form. Here $[g]$ is the conformal class of g.

In the following, we give some applications of the above results. Let

$$\mathcal{C}(M) := \{g \text{ is a Riemannian metric on } M \text{ such that } \sigma_1(g) > 0 \text{ on } M\}$$

be the set of metrics on M with positive scalar curvatures. We define a new invariant

$$\mathcal{A}(M) := \sup_{g \in \mathcal{C}(M)} \mathcal{A}(M, g).$$

Some conformal invariant related to $\mathcal{A}(M, g)$ is studied in [7]. In this work, we do not restrict the study in a conformal class. We want to use the above invariant to study the topology of the manifold M. By the result in [7], we know $\mathcal{A}(\mathbb{S}^3/\Gamma) = \frac{1}{3}$ where Γ is finite subgroup of orthogonal group.

As a direct application of Theorem 1.1, we have the following result.

Corollary 1.5. *Let M be a closed 3-dimensional manifold admitting a Riemannian metric with positive scalar curvature. Suppose that*

$$\mathcal{A}(M) > -\frac{23}{384}.$$

Then M is diffeomorphic to $\mathbb{S}^3/\Gamma$ where Γ is a finite subgroup of orthogonal group.

We conjecture $\mathcal{A}(M)$ could be used to distinguish a sphere and a cylinder. More precisely, we make the following conjecture.

Conjecture 1.6. *When $\mathcal{A}(M) > -1$, then M is diffeomorphic to a spherical space form. When $\mathcal{A}(M) = -1$, then a finite covering of M is diffeomorphic to some cylinder $\mathbb{S}^1 \times \mathbb{S}^2$.*

We could not prove this conjecture right now. However, Theorem 1.3 gives a partial answer.

The paper is organized as follows. In Sec. 2, we recall some definitions and properties on fully nonlinear elliptic PDEs. Then we prove the main results.

2. Proof of Theorems 1.1, 1.2, 1.3 and 1.4

Let (M, g) be a closed 3-dimensional Riemannian manifold with positive scalar curvature. We set

$$S_g^t = \mathrm{Ric}_g - \frac{t}{4} R_g g.$$

It is clear that when $t = 1$, it is just the classical Schouten tensor. Under the conformal change $\widetilde{g} = e^{-2u} g$, we can write

$$S_{\widetilde{g}}^t = S_g^t + \nabla_g^2 u + (1-t)(\triangle_g u) g + du \otimes du - \frac{2-t}{2} |\nabla_g u|_g^2 g.$$

Recall that the positive 2-cone

$$\begin{aligned}\Gamma_2^+ := \{\Lambda = (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{R}^3 \mid \sigma_1(\Lambda) = \lambda_1 + \lambda_2 + \lambda_3 > 0, \\ \sigma_2(\Lambda) = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_1\lambda_3 > 0\}\end{aligned}$$

and consider the subcritical elliptic equation for the metric $\widetilde{g} = e^{-2u} g$ such that

$$(\sigma_2(g^{-1} S_{\widetilde{g}}^t))^{\frac{1}{2}} = f e^{2u}, \tag{2.1}$$

where f is some positive function on M. Such equation is equivalent to

$$\left(\sigma_2\left(g^{-1}\left(S_g^t + \nabla_g^2 u + (1-t)(\triangle_g u) g + du \otimes du - \frac{2-t}{2} |\nabla_g u|_g^2 g\right)\right)\right)^{\frac{1}{2}} = f e^{2u}. \tag{2.2}$$

It is known (see [9]) that when $g^{-1} S_{\widetilde{g}}^t \in \Gamma_2^+$, Eq. (2.2) is elliptic, the linearized equation is a second order linear elliptic PDE. We use the strategy in [9, 2] (see also the parabolic version in [4]) to prove the results. In general, the solution u_t of (2.2) depends on t. We will drop the index t if there is no confusion.

As $R_g > 0$, we could choose $t_0 < 0$ such that $S_g^{t_0}$ is positive definite. We set $f := (\sigma_2(g^{-1} S_g^{t_0}))^{\frac{1}{2}} > 0$. When $t = t_0$, it is clear from the maximum principle that $u \equiv 0$ is the unique solution of (2.2). Now we recall some known results.

Lemma 2.1 (Upper bound Proposition 2.4 in [2]). *Let $u_t \in C^2(M)$ be a solution of equation (2.2) for $t \in [t_0, 2/3]$ with $S^t_{e^{-2u_t}g} \in \Gamma_2^+$. Then there exists a constant $\eta > 0$ independent of t such that*

$$\sup_M u_t \le \eta.$$

Lemma 2.2 ([2, Lemma 3.3]). *Assume $S^t_{\widetilde{g}} \in \Gamma_2^+$ where $\widetilde{g} = e^{-2u}g$. Then there holds*

$$\frac{1}{2}\int_M S^1_g(\nabla_g u, \nabla_g u)dv_g < \frac{3-2t}{8}\int_M R_{\widetilde{g}}|\nabla_g u|^2_g e^{-2u}dv_g + \frac{1}{4}\int_M \Delta_g u|\nabla_g u|^2_g dv_g - \frac{1}{4}\int_M |\nabla_g u|^4_g dv_g.$$

Lemma 2.3 (Gradient estimate [9]). *Let $u_t \in C^3(M)$ be a solution of equation (2.2) for $t \in [t_0, 2/3]$ with $S^t_{e^{-2u_t}g} \in \Gamma_2^+$. Suppose that $\sup_M u_t \le \eta$. Then there exists a constant $\eta_1 > 0$ independent of t such that*

$$\|\nabla_g u_t\|_\infty \le \eta_1.$$

Lemma 2.4 (Integral identity [6]). *Let $\widetilde{g} = e^{-2u}g$ be a conformal metric. Then we have*

$$\int_M \sigma_2(\widetilde{g}^{-1}S^1_{\widetilde{g}})e^{-4u}dv_g = \int_M \sigma_2(g^{-1}S^1_g)dv_g + \frac{1}{8}\int_M R_g|\nabla_g u|^2_g dv_g - \frac{1}{4}\int_M |\nabla_g u|^4_g dv_g + \frac{1}{2}\int_M \triangle_g u|\nabla_g u|^2_g dv_g - \frac{1}{2}\int_M S^1_g(\nabla_g u, \nabla_g u)dv_g.$$

In particular, we have

$$\int_M \sigma_2(\widetilde{g}^{-1}S^1_{\widetilde{g}})e^{-4u}dv_g = \int_M \sigma_2(g^{-1}S^1_g)dv_g + \frac{1}{8}\int_M R_{\widetilde{g}}e^{-2u}|\nabla_g u|^2_g dv_g - \frac{1}{2}\int_M S^1_g(\nabla_g u, \nabla_g u)dv_g.$$

Lemma 2.5 ($C^{2,\alpha}$ estimate [9]). *Let $u_t \in C^4(M)$ be a solution of equation (2.2) for $t \in [t_0, 2/3]$ with $S^t_{e^{-2u_t}g} \in \Gamma_2^+$, verifying $C_1 \le \inf_M u_t \le \sup_M u_t \le C_2$ and $\|\nabla_g u_t\|_\infty \le C_3$ for some constants C_1, C_2, C_3 independent of t. Then for all $\alpha \in (0,1)$, there exists a constant $C_4 > 0$ independent of t such that*

$$\|u_t\|_{C^{2,\alpha}(M,g)} \le C_4.$$

Now we are ready to prove Theorem 1.4.

Proof of Theorem 1.4. First, we prove a lower bound estimate for u_t. For this purpose, we prove the following result.

Claim. *There exists some positive constant $\eta_2 > 0$ independent of t such that $\int_M e^{4u}dv_g \ge \eta_2$.*

For simplicity, we denote $\widetilde{g} = e^{-2u_t} g$, $R = R_g$ and $\widetilde{R} = R_{\widetilde{g}}$. As in [2], we have

$$C \int_M e^{4u} dv_g \geq \int_M \sigma_2(\widetilde{g}^{-1} S^1_{\widetilde{g}}) e^{-4u} dv_g + \frac{1}{16}(1-t)(5-3t) \int_M \widetilde{R}^2 e^{-4u} dv_g$$

for some positive constant C independent of t. Applying Lemmas 2.2 and 2.4, we infer

$$\begin{aligned} C \int_M e^{4u} dv_g \geq& \int_M \sigma_2(g^{-1} S^1_g) dv_g + \frac{1}{8} \int_M \widetilde{R} |\nabla_g u|^2 e^{-2u} dv_g \\ &- \frac{1}{2} \int_M S^1_g(\nabla_g u, \nabla_g u) dv_g + \frac{1}{16}(1-t)(5-3t) \int_M \widetilde{R}^2 e^{-4u} dv_g \\ \geq& \int_M \sigma_2(g^{-1} S^1_g) dv_g - \frac{1-t}{4} \int_M \widetilde{R} |\nabla_g u|^2 e^{-2u} dv_g \\ &- \frac{1}{4} \int_M \Delta_g u |\nabla_g u|^2 dv_g + \frac{1}{4} \int_M |\nabla_g u|^4 dv_g \\ &+ \frac{1}{16}(1-t)(5-3t) \int_M \widetilde{R}^2 e^{-4u} dv_g. \end{aligned}$$

Using the conformal relation $\widetilde{R} e^{-2u} = R + 4\Delta_g u - 2|\nabla_g u|^2$, we get

$$\begin{aligned} C &\int_M e^{4u} dv_g \\ &\geq \int_M \sigma_2(g^{-1} S^1_g) dv_g + \frac{t-1}{4} \int_M (R + 4\Delta_g u - 2|\nabla_g u|^2) |\nabla_g u|^2 dv_g \\ &\quad - \frac{1}{4} \int_M \Delta_g u |\nabla_g u|^2 dv_g + \frac{1}{4} \int_M |\nabla_g u|^4 dv_g \\ &\quad + \frac{1}{16}(1-t)(5-3t) \int_M (R + 4\Delta_g u - 2|\nabla_g u|^2)^2 dv_g \\ &= \int_M \sigma_2(g^{-1} S^1_g) dv_g + \frac{t-1}{4} \int_M R |\nabla_g u|^2 dv_g \\ &\quad + \frac{4t-5}{4} \int_M \Delta_g u |\nabla_g u|^2 dv_g + \frac{3-2t}{4} \int_M |\nabla_g u|^4 dv_g \\ &\quad + \frac{1}{16}(1-t)(5-3t) \int_M (R + 4\Delta_g u - 2|\nabla_g u|^2)^2 dv_g \\ &= \int_M \sigma_2(g^{-1} S^1_g) dv_g + \frac{4t-5}{16} \int_M (R + 4\Delta_g u) |\nabla_g u|^2 dv_g + \frac{1}{16} \int_M R |\nabla_g u|^2 dv_g \\ &\quad + \frac{3-2t}{4} \int_M |\nabla_g u|^4 dv_g + \frac{1}{16}(1-t)(5-3t) \int_M (R + 4\Delta_g u - 2|\nabla_g u|^2)^2 dv_g \end{aligned}$$

$$
\begin{aligned}
&= \int_M \sigma_2(g^{-1}S_g^1)dv_g + \frac{4t-5}{16}\int_M (R+4\Delta_g u - 2|\nabla_g u|^2)|\nabla_g u|^2 dv_g \\
&\quad + \frac{1}{16}\int_M R|\nabla_g u|^2 dv_g + \frac{1}{8}\int_M |\nabla_g u|^4 dv_g \\
&\quad + \frac{1}{16}(1-t)(5-3t)\int_M (R+4\Delta_g u - 2|\nabla_g u|^2)^2 dv_g.
\end{aligned}
$$

Next, we want to estimate

$$
\begin{aligned}
&\frac{4t-5}{16}\int_M (R+4\Delta_g u - 2|\nabla_g u|^2)|\nabla_g u|^2 dv_g + \frac{1}{8}\int_M |\nabla_g u|^4 dv_g \\
&\quad + \frac{1}{16}(1-t)(5-3t)\int_M (R+4\Delta_g u - 2|\nabla_g u|^2)^2 dv_g.
\end{aligned}
$$

For this purpose, we write

$$
\begin{aligned}
&\frac{4t-5}{16}(R+4\Delta_g u - 2|\nabla_g u|^2)|\nabla_g u|^2 \\
&\quad + \frac{1}{8}|\nabla_g u|^4 + \frac{1}{16}(1-t)(5-3t)(R+4\Delta_g u - 2|\nabla_g u|^2)^2 \\
&= \frac{1}{16}(1-t)(5-3t)\left(R+4\Delta_g u + \frac{-25+36t-12t^2}{2(1-t)(5-3t)}|\nabla_g u|^2\right)^2 \\
&\quad + \frac{8t^2-24t+15}{64(1-t)(5-3t)}|\nabla_g u|^4,
\end{aligned}
$$

so that

$$
\begin{aligned}
&\frac{4t-5}{16}\int_M (R+4\Delta_g u - 2|\nabla_g u|^2)|\nabla_g u|^2 dv_g + \frac{1}{8}\int_M |\nabla_g u|^4 dv_g \\
&\quad + \frac{1}{16}(1-t)(5-3t)\int_M (R+4\Delta_g u - 2|\nabla_g u|^2)^2 dv_g \\
&= \frac{1}{16}(1-t)(5-3t)\left(\int_M \left(R+4\Delta_g u + \frac{-25+36t-12t^2}{2(1-t)(5-3t)}|\nabla_g u|^2\right)^2 dv_g \right. \\
&\quad \left. + \int_M \frac{8t^2-24t+15}{4(1-t)^2(5-3t)^2}|\nabla_g u|^4 dv_g\right).
\end{aligned}
$$

By Hölder's inequality, we have

$$
\begin{aligned}
&\int_M \left(R+4\Delta_g u + \frac{-25+36t-12t^2}{2(1-t)(5-3t)}|\nabla_g u|^2\right)^2 dv_g \\
&\quad \geq \frac{1}{\mathrm{vol}(g)}\left(\int_M R dv_g + \frac{-25+36t-12t^2}{2(1-t)(5-3t)}\int_M |\nabla_g u|^2 dv_g\right)^2, \\
&\int_M \frac{8t^2-24t+15}{4(1-t)^2(5-3t)^2}|\nabla_g u|^4 dv_g \geq \frac{1}{\mathrm{vol}(g)}\frac{8t^2-24t+15}{4(1-t)^2(5-3t)^2}\left(\int_M |\nabla_g u|^2 dv_g\right)^2,
\end{aligned}
$$

provided $8t^2 - 24t + 15 \geq 0$. We set

$$A = -25 + 36t - 12t^2, \quad B = 8t^2 - 24t + 15, \quad D = (1-t)(5-3t),$$
$$a = \int_M R dv_g, \quad b = \int_M |\nabla_g u|^2 dv_g.$$

Note that $A^2 + B = 16(t-1)(t-2)(3t-4)(3t-5) = 16D(t-2)(3t-4)$. We note that $A < 0, B > 0, D > 0$ when $t \leq \frac{2}{3}$. We assume that $\min_M R \geq \kappa \frac{1}{\mathrm{vol}(g)} \int_M R dv_g$, so that

$$\int_M R|\nabla_g u|^2 dv_g \geq \kappa \frac{1}{\mathrm{vol}(g)} \int_M R dv_g \int_M |\nabla_g u|^2 dv_g.$$

Then, we get

$$\begin{aligned}
C \int_M e^{4u} dv_g &\geq \int_M \sigma_2(g^{-1} S_g^1) dv_g + \frac{1}{16\,\mathrm{vol}(g)} \kappa ab \\
&\quad + \frac{1}{16\,\mathrm{vol}(g)} D \left(\left(a + \frac{A}{2D} b \right)^2 + \frac{B}{4D^2} b^2 \right) \\
&= \int_M \sigma_2(g^{-1} S_g^1) dv_g + \frac{1}{16\,\mathrm{vol}(g)} \left(D - \frac{D(A+\kappa)^2}{A^2+B} \right) a^2 \\
&\quad + \frac{1}{16\,\mathrm{vol}(g)} \left(\sqrt{\frac{A^2+B}{D}} \frac{b}{2} + \sqrt{\frac{D}{A^2+B}} (A+\kappa) a \right)^2 \\
&\geq \int_M \sigma_2(g^{-1} S_g^1) dv_g + \frac{1}{16\,\mathrm{vol}(g)} \left(D - \frac{D(A+\kappa)^2}{A^2+B} \right) a^2.
\end{aligned}$$

We remark that $\forall\, \kappa \in [0,1]$, $D - \frac{D(A+\kappa)^2}{A^2+B}$ decreases for $t \in (-\infty, \frac{2}{3}]$. Indeed, we calculate

$$\begin{aligned}
\left(D - \frac{D(A+\kappa)^2}{A^2+B} \right)' &= \frac{1}{16(3t^2 - 10t + 8)^2} \Bigg(-(8 + 24\kappa) t^2 \\
&\quad + 6\left(14\kappa + \frac{19}{3} + \kappa^2 \right) t + \left(-76\kappa - \frac{126}{3} - 10\kappa^2 \right) \Bigg),
\end{aligned}$$

which is negative on $(-\infty, \frac{2}{3}]$. Therefore, we infer

$$C \int_M e^{4u} dv_g \geq \int_M \sigma_2(g^{-1} S_g^1) dv_g + \frac{1}{16\,\mathrm{vol}(g)} \left(1 - \frac{(19 - 3\kappa)^2}{384} \right) \left(\int_M R dv_g \right)^2.$$

Therefore, we prove the claim. Together with Lemmas 2.1 and 2.3, u_t is uniformly bounded below. Applying Lemma 2.5, we obtain uniform $C^{2,\alpha}$ bound of u_t. Set

$$\mathfrak{P} := \left\{ t \in \left[t_0, \frac{2}{3}\right] \,\middle|\, \text{there exists } u_t \in C^{2,\alpha} \text{ of (2.2) with } S^t_{e^{-2u_t}g} \in \Gamma_2^+ \right\}.$$

By the standard continuity method, we have $\mathfrak{P} = [t_0, \frac{2}{3}]$ (for detail, see [2]). It follows from the results in [4] and [2] that there exists a metric $\widetilde{g}$ conformal to g such that $S^t_{e^{-2u_t}g} \in \Gamma_2^+$ for $t = \frac{2}{3}$. Thanks for a result in [9, 8], $\mathrm{Ric}_{\widetilde{g}}$ with $\widetilde{g} = e^{-2u_{\frac{2}{3}}} g$ is everywhere positive. Then using a result of Hamilton [12], it follows that M is diffeomorphic to a spherical space form.

Hence, we finish the proof. □

Proof of Theorem 1.1. We see

$$\inf_{\kappa \in (0,1)} \left\{ 1 - \frac{(19 - 3\kappa)^2}{384} \right\} = \frac{23}{384},$$

so that there holds

$$C \int_M e^{4u} dv_g \geq \int_M \sigma_2(g^{-1} S^1_g) dv_g + \frac{1}{16 \operatorname{vol}(g)} \frac{23}{384} a^2 > 0.$$

Hence, we end the proof. □

Proof of Theorem 1.2. By taking $\kappa = 1$, one has

$$C \int_M e^{4u} dv_g \geq \int_M \sigma_2(g^{-1} S^1_g) dv_g + \frac{1}{16 \operatorname{vol}(g)} \frac{1}{3} a^2 > 0.$$

This yields the desired result. □

Remark 2.6. Theorem 1.2 improves the result of Lai in [10].

The proof of Theorem 1.3 is completely different from others. It comes from Theorem 5.7 in [1].

Lemma 2.7. *Let (M^3, g) be a closed Riemannian manifold and let $\bar{\kappa} = \min_M R_g$. If the traceless Ricci tensor satisfies*

$$\|\mathring{\mathrm{Ric}}_g\|^2_{L^2} \leq \frac{1}{6} Y(M, [g])^{3/2} \bar{\kappa}^{1/2}, \tag{2.3}$$

where $Y(M, [g]) = \inf_{\widetilde{g} = e^{-2u}g} \operatorname{vol}(M, \widetilde{g})^{-\frac{1}{3}} \int_M R_{\widetilde{g}} dv_{\widetilde{g}}$ and $\mathring{\mathrm{Ric}}_g = \mathrm{Ric}_g - \frac{1}{3} R_g g$, then

- *either (M, g) is a flat Riemannian metric,*
- *or M is diffeomorphic to a spherical space form,*
- *or (M, g) is isometric to a compact quotient of the Riemannian product $\mathbb{R} \times \mathbb{S}^2$, where $\mathbb{S}^2$ is endowed with a round metric.*

Proof of Theorem 1.3. With help of Lemma 2.7, it is not hard to check that when $Y(M,[g]) = R_g \operatorname{vol}(g)^{2/3}$ the inequality (2.3) is equivalent to

$$\mathcal{A}(M,g) \geq -1.$$

Based on positive constant scalar curvature assumption, we can easily rule out the first case. When the inequality in Lemma 2.7 is strict, the Schrödinger operator

$$\square_g := -\Delta_g + \frac{1}{2}\rho_1$$

is strictly positive, where ρ_1 is the lowest eigenvalue of the Ricci tensor of g. Proposition 3.3 in [1] shows that $b_1(M) = 0$, which rules out the third case in Lemma 2.7. Finally, M is diffeomorphic to a spherical space form. □

Proof of Corollary 1.5. Let g be some Riemannian metric M with positive scalar curvature such that $\mathcal{A}(M,g) > -\frac{23}{384}$. The result follows directly from Theorem 1.1. □

At the end, we ask the following **question:**

Let M and N be 3-dimensional closed manifolds. How to estimate $\mathcal{A}(M\#N)$?

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