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A NOTE ON PRESCRIBED Q-CURVATURE

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A NOTE ON PRESCRIBED Q-CURVATURE

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Let (M, g) be a compact Riemannian manifold of dimension $5 \leq n \leq 7$ with nonnegative scalar curvature and semipositive Q-curvature. Assume (M, g) is not conformally equivalent to the standard sphere. If a prescribed positive smooth function f is flat up to $n-4$ order at some maximum point, there exists a conformal metric $\tilde{g} = u^{4/(n-4)} g$ such that $Q_{\tilde{g}} = f$. This is a natural higher order version of the result of Escobar and Schoen (1986).

1. Introduction

In 1983, Paneitz [2008] introduced a fourth-order conformally invariant operator on an n-dimensional Riemannian manifold, where $n \geq 4$. Not long after, Branson [1985] introduced the Q-curvature. To describe the Paneitz operator, let A denote the Schouten tensor

$$A_g = \frac{1}{n-2}\Big(\mathrm{Ric}_g - \frac{1}{2(n-1)} R_g g\Big),$$

where Ric_g is the Ricci tensor and R_g is the scalar curvature. Let J denotes the trace of A_g, i.e., $J = R_g/(2(n-1))$. Then the Q-curvature is given by

$$Q_g = -\Delta_g J - 2|A_g|^2 + \frac{n}{2} J^2. \tag{1-1}$$

The Paneitz operator is given by

$$P_g u = \Delta_g^2 u + \mathrm{div}_g\{(4A_g - (n-2)Jg)(\nabla u, \cdot)\} + \frac{n-4}{2} Q_g u. \tag{1-2}$$

For $n \geq 5$, suppose $\hat{g} = u^{4/(n-4)} g$ is a conformal metric, the Paneitz operator satisfies

$$P_{\hat{g}} \varphi = u^{-(n+4)/(n-4)} P_g(u\varphi),$$

and then the Q-curvature of $\hat{g}$ is given by

$$Q_{\hat{g}} = \frac{2}{n-4} u^{-(n+4)/(n-4)} P_g u.$$

Due to the conformal invariant of the Paneitz operator, a natural question is: can we find a conformal metric of constant Q-curvature which is a higher order version of

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the Yamabe problem? Recently, fruitful results were obtained, such as [Qing and Raske 2006; Hebey and Robert 2004; Gursky and Malchiodi 2015; Hang and Yang 2016; Gursky et al. 2016; Li and Xiong 2019; Li 2019]. Similar to the positive mass theorem in the Yamabe problem, the analogue in Q-curvature is obtained in [Humbert and Raulot 2009; Gursky and Malchiodi 2015; Hang and Yang 2016]. On the other hand, the prescribed Q-curvature problem also attracts us. Different from the prescribed scalar curvature problem, the lack of maximal principle causes many problems in finding positive solutions. Fortunately, Gursky and Malchiodi [2015] gave the strong maximal principle of P_g under the assumption $R_g \geq 0$ and Q_g is semipositive, which means $Q_g \geq 0$ and $Q_g \not\equiv 0$. The strong maximal principle of P_g plays a crucial role in the existence of positive solutions. Meanwhile, the positivity of P_g is also important and many works are devoted to studying it, such as [Xu and Yang 2001; Djadli et al. 2000; Gursky et al. 2016; Gursky and Malchiodi 2015]. There are some results in special geometric settings. In [Djadli et al. 2000], the authors proved some compactness results in the case where P_g has constant coefficients (for example, (M, g) is an Einstein metric). Subsequently, Esposito and Robert [2002] used the mountain pass theorem to study the related problems. For $n \geq 8$, they gave a sufficient condition of the existence of solutions to the equation

$$P_g u = f|u|^{8/(n-4)}u,$$

for nonlocally conformally flat manifolds but with no information on their sign. Similar perturbation results like the cases about scalar curvature in $\mathbb{S}^n$ were obtained by [Djadli et al. 2002a; 2002b] and others.

Before stating our main results, we give some notation. $Y(M, g)$ denotes the Yamabe invariant. Define a conformal invariant similar to the Yamabe invariant by

$$Y_4(M, g) = \inf_{u \in H^2(M)\setminus\{0\}} \frac{\int_M u P_g u \, \mathrm{d}\mu_g}{\left(\int_M |u|^{2n/(n-4)} \, \mathrm{d}\mu_g\right)^{(n-4)/n}}.$$

When (M, g) is the sphere $\mathbb{S}^n$ with standard metric, the works of [Lieb 1983; Lions 1985a; 1985b; Edmunds et al. 1990] and Proposition 1.1 in [Djadli et al. 2000] show that

$$Y_4(\mathbb{S}^n, g_{\mathbb{S}^n}) = \frac{n(n^2-4)(n-4)\omega_n^{4/n}}{16},$$

where $\omega_n = |\mathbb{S}^n|$. Similarly, to deal with the prescribed scalar curvature problem, we define the functional

$$I_{q,f}(u) = \frac{\int_M u \cdot P_g u \, \mathrm{d}\mu_g}{\left(\int_M f|u|^q \, \mathrm{d}\mu_g\right)^{2/q}}, \tag{1-3}$$

where $2 < q \leq 2n/(n-4)$ and $u \in H^2(M)\setminus\{0\}$. Set $\mu_q = \inf_{u \in H^2(M)\setminus\{0\}} I_{q,f}(u)$.

Inspired by the work of Escobar and Schoen [1986], our main results can be stated as follows:

Theorem 1.1. *Let (M, g) be a compact Riemannian manifold having dimension $n = 5, 6, 7$, or let (M, g) be conformally flat of dimension $n \geq 5$. Assume $R_g \geq 0$, Q_g is semipositive and (M, g) is not conformally equivalent to the standard sphere. Let f be a positive smooth function and flat up to order $n - 4$ at some maximum point. Then there exists a conformal metric $\hat{g} = u^{4/(n-4)} g$ such that $Q_{\hat{g}} = f$.*

Remark 1.2. We notice that the flatness condition holds automatically when $n = 5$. For the nontrivial group G of isometries of $\mathbb{S}^n$, we can get the related results on $\mathbb{S}^n / G$ similar to [Robert 2003].

As a direct consequence, together with the result in [Gursky et al. 2016], there holds:

Theorem 1.3. *Let (M, g) be a compact Riemannian manifold having dimension $n = 6, 7$, or let (M, g) be conformally flat of dimension $n \geq 6$. Assume $Y_4(M, g) > 0$, $Y(M, g) > 0$ and (M, g) is not conformally equivalent to the standard sphere. Let f be a positive smooth function and flat up to order $n - 4$ at some maximum point. Then there exists a conformal metric $\hat{g} = u^{4/(n-4)} g$ such that $Q_{\hat{g}} = f$.*

2. Main proof

Proposition 2.1. *Let (M, g) be a compact Riemannian n-manifold with $n \geq 5$, $R_g \geq 0$ and semipositive Q_g. Let f be a smooth positive function on M. If*

$$\mu_{2n/(n-4)} < \frac{Y_4(\mathbb{S}^n, g_{\mathbb{S}^n})}{(\max f)^{(n-4)/n}}, \tag{2-1}$$

then there exists a positive $u \in C^\infty(M)$ such that f be the Q-curvature of the conformal metric $\bar{g} = u^{4/(n-4)} g$.

Proof. Based on the condition $R_g \geq 0$ and Q_g is semipositive, we apply Theorem 2.2 in [Gursky and Malchiodi 2015] to conclude that P_g satisfies the strong maximum principle, i.e., for all $u \in C^4(M)$, if $P_g u \geq 0$, then either $u > 0$ or $u \equiv 0$ on M. Meanwhile, Proposition 2.3 in [Gursky and Malchiodi 2015] shows that P_g is coercive. With these two important properties, combine Proposition 4.1 with Theorem 5.2 in [Robert 2009] to get the desired result. For convenience, we give a brief proof here. For any $q \in (2, 2n/(n - 4))$, with the compactness of the embedding $H^2(M) \hookrightarrow L^q(M)$ and the fact that P_g is coercive, use the standard variational method to get $u_q \in H^2(M) \backslash \{0\}$ which achieves the minimum of $I_{q,f}$ and weakly solves

$$P_g u_q = \mu f |u_q|^{q-2} u_q, \tag{2-2}$$

where $\mu = \mu_q\big(\int_M f|u_q|^q \,\mathrm{d}\mu_g\big)^{2/q-1}$. As for the regularity of the weak solution, different from elliptic cases, we need the techniques developed in [Van der Vorst 1993; Djadli et al. 2000] and Proposition 3 in [Esposito and Robert 2002] to get $u_q \in C^4(M)$, which is then a strong solution of (2-2). Let $\tilde{u}_q$ solve the equation $P_g\tilde{u}_q = |P_g u_q|$. Notice that $|P_g u_q| \in C^{0,1}(M)$, and then apply the regularity result in Proposition 3 of [Esposito and Robert 2002] to get $\tilde{u}_q \in C^4(M)$. On one hand, due to the strong maximum principle of P_g, there holds $\tilde{u}_q > 0$. On the other hand, since $P_g(\tilde{u}_q \pm u_q) \geq 0$, there holds $\tilde{u}_q \geq |u_q|$. By a direct computation, there holds

$$\begin{aligned} I_{q,f}(\tilde{u}_q) = \frac{\int_M \tilde{u}_q P_g \tilde{u}_q \,\mathrm{d}\mu_g}{\big(\int_M f|\tilde{u}_q|^q \,\mathrm{d}\mu_g\big)^{2/q}} &= \frac{\mu \int_M \tilde{u}_q f |u_q|^{q-1} \,\mathrm{d}\mu_g}{\big(\int_M f|\tilde{u}_q|^q \,\mathrm{d}\mu_g\big)^{2/q}} \\ &\leq \mu\Big(\int_M f\tilde{u}_q^q \,\mathrm{d}\mu_g\Big)^{-1/q} \cdot \Big(\int_M f|u_q|^q \,\mathrm{d}\mu_g\Big)^{1-1/q} \\ &\leq \mu\Big(\int_M f|u_q|^q \,\mathrm{d}\mu_g\Big)^{1-2/q} = \mu_q. \end{aligned}$$

Thus, $\tilde{u}_q$ also achieves the minimum of the energy and solves the equation

$$P_g\tilde{u}_q = \tilde{\mu} f \tilde{u}_q^{q-1}, \tag{2-3}$$

where $\tilde{\mu} = \mu_q\big(\int_M f\tilde{u}_q^q \,\mathrm{d}\mu_g\big)^{2/q-1}$. With the restriction (2-1), for $q = 2n/(n-4)$, we obtain a weak solution $u \neq 0$ such that

$$P_g u = f|u|^{8/(n-4)}u,$$

which is a minimizer of $I_{2n/(n-4),f}$. By the regularity result in Proposition 3 of [Esposito and Robert 2002], there holds $u \in C^4(M)$. Take the same strategy as above to obtain the positive minimizer $u^* \in C^4(M)$ of the functional $I_{(2n)/n-4,f}(u)$. Due to f being smooth, the higher regularity of u^* can be obtained. We refer to [Robert 2009] for more details. □

Proposition 2.2. *Let (M, g) be a compact Riemannian manifold having dimension $n = 5, 6, 7$, or let (M, g) be conformally flat of dimension $n \geq 5$. Assume $R_g \geq 0$ and Q_g is semipositive. Let f be a positive smooth function and flat up to order $n-4$ at some maximum point x_0. If (M, g) is not conformally equivalent to the standard sphere, then for $\varepsilon > 0$ small, there exists a function $\psi_\varepsilon \in C^\infty(M)$ and a constant $c_{x_0} > 0$ such that*

$$I_{2n/(n-4),f}(\psi_\varepsilon) \leq \frac{Y_4(\mathbb{S}^n, g_{\mathbb{S}^n})}{(\max f)^{(n-4)/n}} - c_{x_0}\varepsilon^{n-4}. \tag{2-4}$$

Proof. Based on Proposition 2.5 in [Gursky and Malchiodi 2015], consider the conformal normal coordinates $\tilde{g} = \varphi^{4/(n-4)}g$ of [Lee and Parker 1987] at x_0 under our assumption. The Green's function for the Paneitz operator with pole has the following expansion:

$$G_{x_0}(x) = \frac{c_n}{d_{\tilde g}(x, x_0)^{n-4}} + \alpha_{x_0} + O^{(4)}(r),$$

where $c_n = 1/((n-2)(n-4)\omega_{n-1})$, $\omega_{n-1} = |\mathbb{S}^{n-1}|$ and $O^{(k)}(r^m)$ denotes any quantity ϕ satisfying $|\nabla^j \phi(x)| \le C_j r^{m-j}$ for $1 \le j \le k$, where $r = |x| = d_{\tilde g}(x, x_0)$. Under our assumption, there holds $\alpha_{x_0} > 0$ by Theorem 2.9 in [Gursky and Malchiodi 2015]. Let $\eta(x)$ be a cut-off function with support in a ball $B_{2\delta}(x_0)$ and equal to 1 in $B_\delta(x_0)$. Set

$$u_\varepsilon(x) = \frac{\eta(x)}{\left(\varepsilon^2 + d_{\tilde g}(x, x_0)^2\right)^{(n-4)/2}}.$$

As in Lemma 4.2 of [Gursky and Malchiodi 2015], there holds

$$P_{\tilde g}(u_\varepsilon) = \frac{b_n \varepsilon^4}{(\varepsilon^2 + |x|^2)^{(n+4)/2}} + \frac{O(1)}{(\varepsilon^2 + |x|^2)^{(n-4)/2}},$$

where $b_n = n(n-4)(n^2-4)$. Consider the function $\hat u_\varepsilon$ defined by

$$P_{\tilde g} \hat u_\varepsilon = \frac{\eta(x) b_n \varepsilon^4}{(\varepsilon^2 + |x|^2)^{(n+4)/2}}.$$

We consider a cut-off function $\tilde\chi_{\tilde\delta}(x) = \tilde\chi(x/\tilde\delta)$, where $\tilde\chi$ is a cut-off function equal to 1 in B_1 and equal to zero outside B_2. We define an approximate solution $\check u_\varepsilon$ by

$$\check u_\varepsilon = \tilde\chi_{\tilde\delta}(u_\varepsilon + \beta) + (1 - \tilde\chi_{\tilde\delta}) \frac{1}{c_n} G_{x_0},$$

with $\tilde\delta \ll \delta$, which is positive. By Lemma 5.3 in [Gursky and Malchiodi 2015], $|\hat u_\varepsilon - \check u_\varepsilon| = o_{\tilde\delta}(1)$. By the direct computation as in Section 5 of [Gursky and Malchiodi 2015], one has

$$\text{(2-5)} \quad \int_M \hat u_\varepsilon P_{\tilde g} \hat u_\varepsilon \, \mathrm{d}\mu_{\tilde g} = b_n \varepsilon^4 \Big(\int_M u_\varepsilon^{2n/(n-4)} \, \mathrm{d}\mu_{\tilde g} + \beta(1 + o(1)) \int_M u_\varepsilon^{(n+4)/(n-4)} \, \mathrm{d}\mu_{\tilde g} + O(1) \Big),$$

where $\beta = \alpha_{x_0}/c_n > 0$.

Due to the flatness condition of f at x_0, we have $f(x_0) - f(x) \le c|x|^{n-3}$ in $B_{\tilde\delta}(x_0)$. Then we have the following estimate:

$$\begin{aligned}\int_M f |\hat u_\varepsilon|^{2n/(n-4)} \, \mathrm{d}\mu_{\tilde g} &= \int_{B_{\tilde\delta}(x_0)} f(x)(u_\varepsilon + \beta)^{2n/(n-4)} \, \mathrm{d}\mu_{\tilde g} + O(1) \\ &\ge f(x_0) \int_{B_{\tilde\delta}(x_0)} (u_\varepsilon + \beta)^{2n/(n-4)} \, \mathrm{d}\mu_{\tilde g} - c \int_{B_{\tilde\delta}(x_0)} |x|^{n-3} (u_\varepsilon + \beta)^{2n/(n-4)} \, \mathrm{d}\mu_{\tilde g} + O(1).\end{aligned}$$

Since in $B_{\tilde\delta}(x_0)$, β is bounded by u_ε, we have

$$\begin{aligned}\int_M (u_\varepsilon+\beta)^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g} &= \int_{B_{\tilde\delta}(x_0)} u_\varepsilon^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g}+\frac{2n}{n-4}\beta\int_{B_{\tilde\delta}(x_0)} u_\varepsilon^{(n+4)/(n-4)}\,\mathrm{d}\mu_{\tilde g} \\ &\qquad +\beta^2\int_{B_{\tilde\delta}(x_0)} O(u_\varepsilon^{8/(n-4)})\,\mathrm{d}\mu_{\tilde g}+O(1) \\ &= \int_M u_\varepsilon^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g}+\frac{2n}{n-4}\beta\int_M u_\varepsilon^{(n+4)/(n-4)}\,\mathrm{d}\mu_{\tilde g}+O(\varepsilon^{n-8})+O(1) \\ &= c_1\varepsilon^{-n}+c_2\varepsilon^{-4}+O(\varepsilon^{n-8})+O(1)\end{aligned}$$

and

$$\begin{aligned}\int_{B_{\tilde\delta}(x_0)} |x|^{n-3}(u_\varepsilon+\beta)^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g} &\le C\int_{B_{\tilde\delta}(x_0)} |x|^{n-3}u_\varepsilon^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g} \\ &\le C\int_{B_{\tilde\delta}} |x|^{n-3}(\varepsilon^2+|x|^2)^{-n}\,\mathrm{d}x = O(\varepsilon^{-3}).\end{aligned}$$

Thus due to the fact that $n-8\ge -3$, there holds

$$\begin{aligned}\int_M f|\hat u_\varepsilon|^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g} \\ \ge f(x_0)\Big(\int_M u_\varepsilon^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g}+\frac{2n}{n-4}\beta\int_M u_\varepsilon^{(n+4)/(n-4)}\,\mathrm{d}\mu_{\tilde g}+O(\varepsilon^{-3})\Big).\end{aligned}\tag{2-6}$$

Combining estimates (2-5) and (2-6), a Taylor expansion implies

$$\begin{aligned}I_{2n/(n-4),f}(\varphi\hat u_\varepsilon) &= \frac{\int_M \hat u_\varepsilon P_{\tilde g}\hat u_\varepsilon\,\mathrm{d}\mu_{\tilde g}}{\big(\int_M f|\hat u_\varepsilon|^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g}\big)^{(n-4)/n}} \\ &\le \frac{Y_4(S_n)}{f(x_0)^{(n-4)/n}}\left(1-\beta(1+o(1))\frac{\int_M u_\varepsilon^{2n/(n-4)}\,\mathrm{d}\mu_{\tilde g}}{\int_M u_\varepsilon^{(n+4)/(n-4)}\,\mathrm{d}\mu_{\tilde g}}\right) \\ &\le \frac{Y_4(S_n)}{f(x_0)^{(n-4)/n}}-c_{x_0}\varepsilon^{n-4}.\end{aligned}$$ □

Proofs of Theorem 1.1 and Theorem 1.3. Combining together Proposition 2.1 with Proposition 2.2, we prove Theorem 1.1. Under the assumption on (M,g) of Theorem 1.3, we get a conformal metric $g_1=u_1^{4/(n-4)}g$ such that $R_{g_1}>0$ and $Q_{g_1}>0$ by using Theorem 1.1 in [Gursky et al. 2016]. Apply Theorem 1.1 to get a conformal metric $g_2=u_2^{4/(n-4)}g_1=(u_1u_2)^{4/(n-4)}g$ such that $Q_{g_2}=f$. □

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