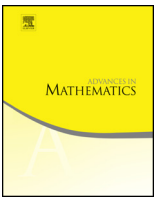
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The total Q-curvature, volume entropy and polynomial growth polyharmonic functions



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ABSTRACT

In this paper, we investigate a conformally flat and complete manifold $(M, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ with finite total Q-curvature. We introduce a new volume entropy, incorporating the background Euclidean metric, and demonstrate that the metric g is normal if and only if the volume entropy is finite. Furthermore, we establish an identity for the volume entropy utilizing the integrated Q-curvature. Additionally, under normal metric assumption, we get a result concerning the behavior of the geometric distance at infinity compared with Euclidean distance. With help of this result, we prove that each polynomial growth polyharmonic function on such manifolds is of finite dimension. Meanwhile, we prove several rigidity results by imposing restrictions on the sign of the Q-curvature. Specifically, we establish that on such manifolds, the Cohn-Vossen inequality achieves equality if and only if each polynomial growth polyharmonic function is a constant.

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1. Introduction

When considering a complete surface (M^2, g), the famous Chern-Gauss-Bonnet formula was extended to non-compact surfaces by Cohn-Vossen [18] and Huber [27]. Specifically, if the Gaussian curvature K is absolutely integrable, then the formula states that:

$$\int_M K \mathrm{d}v_g \leq 2\pi\chi(M) \tag{1.1}$$

where $\chi(M)$ is the Euler number of M. Notably, in [27], Huber demonstrated that such a surface M can be conformally transformed into a closed surface with a finite number of punctures. To explore the generalized version of Huber's theorem, we refer the interested reader to [12], [9], [36] and the relevant references therein. Additionally, Finn [25] established that the deficit is a sum of non-negative isoperimetric ratios ν_i near infinity, expressed as follows:

$$\chi(M) - \frac{1}{2\pi}\int_M K \mathrm{d}v_g = \sum \nu_i. \tag{1.2}$$

In higher-dimensional cases, the Q-curvature performs a role similar to the Gaussian curvature. Naturally, one may wonder if analogous results exist for higher dimensions. For this situation, Chang, Qing, and Yang ([11], [12]) achieved a significant breakthrough. Before presenting their results, we provide some background information for reader's convenience.

During the 1980s, Paneitz [41] introduced a renowned conformal operator denoted by

$$P_g^4 = \Delta_g^2 - \mathrm{div}_g\left((\frac{2}{3}R_g g - 2Ric_g)d\right)$$

where Δ_g represents the Laplace-Beltrami operator, and R_g together with Ric_g denote the scalar curvature and Ricci curvature tensor of (M^4, g), respectively. Subsequently, Branson [5] introduced the Q-curvature (up to scaling) defined by

$$Q_g^4 = -\frac{1}{6}(\Delta_g R_g - R_g^2 + 3|Ric_g|^2).$$

Remarkably, the Q-curvature satisfies the conformal invariant equation similar to Gaussian curvature:

$$P_g^4 u + Q_g^4 = Q_{\tilde{g}}^4 e^{4u}$$

for the conformal metric $\tilde{g} = e^{2u}g$. For higher order cases, Graham, Jenne, Mason and Sparling [26] introduced an operator known as GJMS operator P_g^n for (M^n, g) satisfying

$$P_g^n u + Q_g^n = Q_{\tilde{g}}^n e^{nu}$$

where $\tilde{g} = e^{2u} g$ and $n \geq 4$ is an even integer. Numerous outstanding results have been obtained for compact manifolds and for further details, we refer to [13], [6], [22] and the references therein.

In this paper, we focus on conformally flat cases, i.e. those of the form $(M, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$, where $n \geq 2$ is an even integer and the Q-curvature Q_g satisfying the equation:

$$(-\Delta)^{\frac{n}{2}} u = Q_g e^{nu}. \tag{1.3}$$

For the case $n = 2$, the Q-curvature Q_g is exactly the Gaussian curvature. Meanwhile, when $n \geq 4$, we have the scalar curvature R_g satisfying

$$R_g = 2(n-1)e^{-2u}(-\Delta u - \frac{n-2}{2}|\nabla u|^2). \tag{1.4}$$

This equation (1.3) holds significant importance across various fields. Of particular note, when $Q(x) = 1$ and the finite volume assumption i.e. $e^{nu} \in L^1(\mathbb{R}^n)$ holds, classification theorems outlined in works such as [15], [34], [48], [51] and [39] are highly instrumental in tackling many related problems. We define a conformally flat manifold $(\mathbb{R}^n, e^{2u}|dx|^2)$ to have finite total Q-curvature if

$$\int_{\mathbb{R}^n} |Q_g| e^{nu} \mathrm{d}x < +\infty. \tag{1.5}$$

For simplicity, we set a normalized integrated Q-curvature as

$$\alpha_0 := \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} Q_g e^{nu} \mathrm{d}x$$

throughout this paper where $|\mathbb{S}^n|$ denotes the volume of standard sphere in $\mathbb{R}^{n+1}$.

The assumption of finite total Q-curvature is particularly useful as it allows us to represent solutions to (1.3) as logarithmic potentials known as normal solutions in certain cases. The solution $u(x)$ to (1.3) is said to be normal if it satisfies the following integral equation

$$u(x) = \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} Q_g(y) e^{nu(y)} \mathrm{d}y + C \tag{1.6}$$

where C is a constant. We will discuss normal solutions in details in the next section.

Normal solutions play a crucial role in various works, including [25], [27], [11], [12], and many others. Furthermore, if u is a normal solution, the conformal metric $g = e^{2u}|dx|^2$

is considered as a normal metric. Throughout this paper, we just focus on the smooth metric $g = e^{2u}|dx|^2$. For general complete locally conformally flat manifolds with simple ends, we will discuss in our forthcoming paper.

For the sake of convenience, we use the notation $B_R(p)$ to refer to an Euclidean ball in $\mathbb{R}^n$ centered at $p \in \mathbb{R}^n$ with a radius of R. Moreover, we use $V_g(\cdot)$ to denote the measure with respect to the metric $g = e^{2u}|dx|^2$, and $d_g(\cdot,\cdot)$ represents the geodesic distance. Also, $|B_R(p)|$ refers to the volume of $B_R(p)$ in terms of the Euclidean metric. For a function $\varphi(x)$, the positive part of $\varphi(x)$ is denoted as $\varphi(x)^+$ and the negative part of $\varphi(x)$ is denoted as $\varphi(x)^-$. Set $⨍_E \varphi(x)\mathrm{d}x = \frac{1}{|E|}\int_E \varphi(x)\mathrm{d}x$ for any measurable set E. For a constant C, C^+ denotes C if $C \geq 0$, otherwise, $C^+ = 0$. Here and thereafter, we denote by C a constant which may be different from line to line. For $s \in \mathbb{R}$, $[s]$ denotes the largest integer not greater than s.

In their work [11], Chang, Qing and Yang derived a Chern-Gauss-Bonnet formula for a complete manifold $(\mathbb{R}^4, e^{2u}|dx|^2)$ generalizing Cohn-Vossen inequality (1.1) and Fin's identity (1.2) to higher dimensional manifolds with help of Q-curvature. Supposing $g = e^{2u}|dx|^2$ is a complete metric on $\mathbb{R}^4$ with finite total Q-curvature and $R_g \geq 0$ near infinity, then there holds

$$1 - \frac{1}{8\pi^2}\int\limits_{\mathbb{R}^4} Q_g e^{4u}\mathrm{d}x = \lim_{R\to\infty} \frac{V_g(\partial B_R(0))^{\frac{4}{3}}}{4(2\pi^2)^{\frac{1}{3}} V_g(B_R(0))}. \tag{1.7}$$

In fact, the results hold for higher order cases as well, and references such as [12], [24], and [40] can provide more information. It's worth noting that the right side of the aforementioned identity represents an isoperimetric ratio. Bonk, et al. [7] and Wang ([46], [47]) have studied the isoperimetric inequality on such manifolds, under additional assumptions on the value of the integrated Q-curvature, using the properties of the strong A_∞ weight, which will be discussed in Section 3. Similar results can also be found in [31], [1], [50] and the references therein.

In several notable works, including [38], [30], and the references therein, the concept of "volume entropy" has been utilized to characterize various properties of manifolds. We also recommend interested readers to refer to a nice survey [8] for more detailed information about volume growth on complete manifolds. In this paper, we introduce a new volume entropy $\tau(g)$ associated with the conformal background metric, which differs from the definition given in [38]. The volume entropy $\tau(g)$ of the metric $g = e^{2u}|dx|^2$ is defined as follows:

$$\tau(g) := \lim_{R\to\infty} \sup \frac{\log V_g(B_R(0))}{\log |B_R(0)|}. \tag{1.8}$$

We say the volume entropy of the metric g is finite if $\tau(g) < +\infty$. As mentioned earlier, the normal metric plays a crucial role in dealing with conformally flat manifolds. Unlike the approach in [11], which relies on restrictions regarding the sign of scalar curvature near infinity, our first result employs the concept of volume entropy. By utilizing volume

entropy, we establish a necessary and sufficient condition for a complete metric with finite total Q-curvature to be classified as a normal metric.

Theorem 1.1. *Consider a conformally flat and complete manifold* $(M, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ *with finite total Q-curvature where* $n \geq 2$ *is an even integer.*

(i) The metric g *is normal if and only if the volume entropy* $\tau(g)$ *is finite.*
(ii) If the volume entropy $\tau(g)$ *is finite, there holds*

$$\tau(g) = 1 - \alpha_0. \tag{1.9}$$

Remark 1.2. In [27], for the two-dimensional case, the finite total Gaussian curvature assumption and the completeness of the metric are already sufficient to ensure the normal metric. However, for higher dimensional cases, the finite total Q-curvature is not enough to ensure that (refer to Remark 1.2 in [11]), mainly because the complexity of the kernel of P_g. Besides, the completeness of metric is also crucial since there exist non-normal solutions with finite total Q-curvature and finite volume entropy (See [10], [29], [49]). In [53], Zhang showed that the metric is normal under the assumption on the decay rate of Q-curvature near infinity.

Corollary 1.3. *Consider a conformally flat and complete manifold* $(M, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ *with finite total Q-curvature where* $n \geq 2$ *is an even integer. If the volume entropy* $\tau(g)$ *is finite and* $Q_g \geq 0$*, then*

$$\tau(g) \leq 1$$

with equality holds if and only if $u \equiv C$*.*

In Li and Tam's work [31], they focus on a key geometric result concerning the behavior of geodesic distance at infinity when compared to a flat metric background. Their main objective is to prove Yau's conjecture [52] regarding the dimensions of polynomial growth of harmonic functions. For more detailed information, interested readers can refer to [32], [19], [20], and [14]. For higher dimensional cases, we also obtain a distance comparison identity under the normal metric assumption.

Theorem 1.4. *Consider a conformally flat manifold* $(M, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ *with finite total Q-curvature where* $n \geq 2$ *is an even integer. Supposing that the metric* g *is normal, then for each fixed point* p*, there holds*

$$\lim_{|x|\to\infty} \frac{\log d_g(x, p)}{\log |x - p|} = (1 - \alpha_0)^+ .$$

Moreover, if $\alpha_0 > 1$*, one has*

$$\operatorname{diam}(\mathbb{R}^n, e^{2u}|dx|^2) < +\infty$$

where the diameter is defined by $\operatorname{diam}(M,g) := \sup_{x,y\in M} d_g(x,y)$.

In [11], Chang, Qing and Yang demonstrated that if the metric is normal and complete, then $\alpha_0 \leq 1$. It is interesting to consider whether we can provide a restriction on α_0 and deduce the completeness of the metric. In the case of two dimensions, Aviles provided a result (see Theorem C in [3]). Here, we can utilize Theorem 1.4 to demonstrate that the metric is complete if $\alpha_0 < 1$.

Corollary 1.5. *Consider a conformally flat manifold* $(M,g) = (\mathbb{R}^n, e^{2u}|dx|^2)$ *with finite total Q-curvature where* $n \geq 2$ *is an even integer. Suppose that the metric* g *is normal. If* $\alpha_0 < 1$*, then the metric* g *is complete.*

Remark 1.6. If $\alpha_0 = 1$, the situation becomes more subtle. The metric can be either complete or non-complete. In Section 6, we will construct some examples to confirm this.

In the context of higher-dimensional manifolds $(\mathbb{R}^n, e^{2u}|dx|^2)$, it is natural to consider the kernel of the GJMS operator P_g on such manifolds and we refer to these kernel functions as polyharmonic functions. Precisely, for $d \geq 0$, we define $\mathcal{PH}_d(M,g)$ as the linear space of the kernel function of P_g of polynomial growth at most d: for some fixed point $p \in M$

$$\mathcal{PH}_d(M,g) := \{f(x) | P_g f(x) = 0, |f(x)| \leq C(d_g(x,p)^d + 1)\}.$$

Similarly, define the polynomial growth polyharmonic functions on $\mathbb{R}^n$ respect to standard Euclidean metric as follows:

$$\mathcal{PH}_d(\mathbb{R}^n, |dx|^2) := \{f(x) | (-\Delta)^{\frac{n}{2}} f = 0, |f(x)| \leq C(|x|^d + 1)\}.$$

With help of a Liouville-type theorem, it is well-known that such linear spaces consists of polynomials (Seeing [4], [2]) and the dimension of such linear space is finite for each $d \geq 0$.

Taking inspiration from [31] in the case of two dimensional surfaces, our objective is to establish a framework for analyzing the linear spaces $\mathcal{PH}_d(M,g)$ of polynomial growth polyharmonic functions on conformally flat manifolds. It is important to note that the methodology employed in [31] may not be directly applicable to higher dimensional scenarios. Instead, our current approach builds upon the foundation laid in our previous work in [33]. Similar to Theorem 4.6 in [31], if Q_g is non-negative outside a compact set, we will obtain an identity. Moreover, if $Q_g \geq 0$, we are able to derive a rigidity result similar to the findings presented in [14] for manifolds with non-negative Ricci curvature. Through the utilization of Fin's identity (1.2) and (1.7), we know that the

isoperimetric ratio vanishes when the Cohn-Voseen inequality (1.1) achieves the identity. Here, we would like to provide an alternative characterization using polynomial growth polyharmonic functions.

Theorem 1.7. *Consider a conformally flat and complete manifold* $(M, g) = (\mathbb{R}^n, e^{2u}|dx|^2)$, *where* $n \geq 2$ *is an even integer, and the total Q-curvature is finite. Assume that the volume entropy* $\tau(g)$ *is finite. Then the following results are established:*

(i) For each $d \geq 0$, *there holds*

$$\dim\left(\mathcal{PH}_d(M, g)\right) \leq \dim(\mathcal{PH}_{d\tau(g)}(\mathbb{R}^n, |dx|^2)).$$

(ii) If Q_g *is non-negative outside a compact set and* $d \geq 0$, *there holds*

$$\dim(\mathcal{PH}_d(M, g)) = \dim(\mathcal{PH}_{d\tau(g)}(\mathbb{R}^n, |dx|^2)). \tag{1.10}$$

(iii) If $Q_g \geq 0$ *everywhere, for each integer* $k \geq 1$,

$$\dim(\mathcal{PH}_k(M, g)) \leq \dim(\mathcal{PH}_k(\mathbb{R}^n, |dx|^2))$$

with equality holds if and only if $u \equiv C$.

(iv) The volume entropy $\tau(g) = 0$ *if and only if for each* $d \geq 0$, $\mathcal{PH}_d(M, g) =$ {constant functions}.

Here is a brief overview of the structure of this paper. In Section 2, we establish some properties of the logarithmic potential and give a decomposition result in Theorem 2.2 under the assumption of finite volume entropy. We also provide an estimate for the volume entropy in Theorem 2.14. As a practical application, we get a lower bound of the integrated Gaussian curvature by considering a finite volume assumption instead of the completeness of the metric in Theorem 2.16 and Theorem 2.18. In Section 3, we discuss some properties of strong A_∞ weights and give the proof of Theorem 1.4. Meanwhile, we derive a volume and length comparison identity in Theorem 3.2 if the metric is normal and complete. In Section 4, we introduce a criterion for the solution (1.3) to be normal solutions in Theorem 4.2 related to scalar curvature. Meanwhile, we give the proofs of Theorem 1.1 and Corollary 1.3. In Section 5, we prove Theorem 1.7 with help of the geodesic distance comparison identity in Theorem 3.2. Finally, in Section 6, we provide some examples to help illustrate the subtle case when $\alpha_0 = 1$.

2. Asymptotic behavior of logarithmic potential

To enhance our understanding of the logarithmic potential on non-compact manifolds, it is helpful to first discuss Green's representation on compact manifolds. On a four-dimensional compact manifold (M, g_0), if the kernel of the Paneitz operator P_{g_0} consists

of constants, the Green's function $G_{g_0}(x,y)$ exists (see Lemma 1.7 in [13]), and we have a nice representation of the solution to the equation $P_{g_0}u = \varphi(x)$ as

$$u(x) - \frac{1}{V_{g_0}(M)} \int\limits_M u \mathrm{d}\mu_{g_0} = \int\limits_M G_{g_0}(x,y)\varphi(y)\mathrm{d}\mu_{g_0}.$$

Equivalently, we can rewrite it as

$$u(x) = \int\limits_M (G_{g_0}(x,y) - G_{g_0}(x_0,y))\varphi(y)\mathrm{d}\mu_{g_0} + u(x_0). \tag{2.1}$$

Returning to our current setting, we note that on $\mathbb{R}^n$, the kernel of the polyharmonic Laplacian operator $(-\Delta)^{\frac{n}{2}}$ contains more than just constants, which brings significant difficulties to our research. Consider the following equation for even integer $n \geq 2$ given by the equation

$$(-\Delta)^{\frac{n}{2}} w(x) = f(x), \tag{2.2}$$

where $w(x)$ and $f(x)$ are both smooth functions on $\mathbb{R}^n$ with $f \in L^1(\mathbb{R}^n)$. We aim to investigate whether the solutions to (2.2) can be represented analogously to (2.1) under certain assumptions. Actually, the Green's function of the polyharmonic Laplacian operator $(-\Delta)^{\frac{n}{2}}$ satisfying (See Chapter 1.2 in [2])

$$(-\Delta)^{\frac{n}{2}} \left(\frac{2}{(n-1)!|\mathbb{S}^n|} \log \frac{1}{|x|} \right) = \delta_0(x) \tag{2.3}$$

where $\delta_0(x)$ is the Dirac operator. Compared with (2.1), we naturally define a logarithmic potential as follows:

$$\mathcal{L}(f)(x) = \frac{2}{(n-1)!|\mathbb{S}^n|} \int\limits_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} f(y)\mathrm{d}y.$$

Since smooth $f \in L^1(\mathbb{R}^n)$, it is not hard to check that the above logarithmic potential is well-defined. Our definition of the logarithmic potential differs slightly from a similar definition introduced in [7]. We should point out that the logarithmic potential has been previously introduced and studied in various forms and special cases by many researchers, including in [15], [34], and other works.

Definition 2.1. We say $w(x)$ is a normal solution to (2.2) if for some constant C,

$$w(x) = \mathcal{L}(f)(x) + C.$$

We introduce a quantity like the volume entropy $\tau(g)$ to measure the volume growth of $w(x)$ on $\mathbb{R}^n$, denoted as $\mathcal{V}_{sup}(e^{nw})$ and defined by the formula:

$$\mathcal{V}_{sup}(e^{nw}) := \lim_{R\to\infty} \sup \frac{\log \int_{B_R(0)} e^{nw} \mathrm{d}x}{\log |B_R(0)|}.$$

Similarly, we define $\mathcal{V}_{inf}(e^{nw})$ as:

$$\mathcal{V}_{inf}(e^{nw}) := \lim_{R\to\infty} \inf \frac{\log \int_{B_R(0)} e^{nw} \mathrm{d}x}{\log |B_R(0)|}.$$

If $\mathcal{V}_{sup}(e^{nw}) < +\infty$ with $\mathcal{V}_{sup}(e^{nw}) = \mathcal{V}_{inf}(e^{nw})$, we say that the limit of volume growth exists and denote it as $\mathcal{V}(e^{nw})$. The function $w(x)$ is said to have polynomial volume growth if $\mathcal{V}_{sup}(e^{nw})$ is finite. In fact, one may verify that $\mathcal{V}_{sup}(e^{nw})$ is finite if and only if there exists $s \geq 0$ such that

$$\int_{B_R(0)} e^{nw} \mathrm{d}x = O(R^s).$$

Assuming polynomial volume growth of $w(x)$, we can decompose the solution $w(x)$ of equation (2.2) into the logarithmic potential and a polynomial. Furthermore, we can derive a necessary and sufficient condition for being normal solutions for (2.2), which is of particular interest.

Theorem 2.2. *Let $n \geq 2$ be an even integer and consider a smooth solution $w(x)$ to (2.2) with smooth $f \in L^1(\mathbb{R}^n)$. Suppose that $w(x)$ has polynomial volume growth. Then we have a decomposition*

$$w(x) = \mathcal{L}(f)(x) + P(x) \tag{2.4}$$

where $P(x)$ is a polynomial with degree at most $n-2$ and $P(x) \leq C \log(|x|+2)$.

Moreover, for $n \geq 4$, $P(x) \equiv C$ if and only if one of the following conditions holds

(a)

$$\int_{B_R(0)} |\Delta w| \mathrm{d}x = o(R^n). \tag{2.5}$$

(b)

$$\int_{B_R(0)} |w| \mathrm{d}x = o(R^{n+2}). \tag{2.6}$$

Before establishing the proof of Theorem 2.2, we need introduce some helpful lemmas. For brevity, set the notation α defined as

$$\alpha = \frac{2}{(n-1)!|\mathbb{S}^n|}\int\limits_{\mathbb{R}^n} f(x)\mathrm{d}x.$$

It is natural to expect that the logarithmic potential $\mathcal{L}(f)(x)$ has an asymptotic behavior near infinity resembling $-\alpha\log|x|$. In previous works such as [17] and [16], such behavior has been described under additional decay assumptions to obtain pointwise estimates. In the current paper, we will obtain integral results without requiring any additional assumptions.

Lemma 2.3. *Given $f(x)$ satisfying $f \in L^\infty_{loc}(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$ with even integer $n \geq 2$, for $|x| \gg 1$, there holds*

$$\mathcal{L}(f)(x) = (-\alpha + o(1))\log|x| + \frac{2}{(n-1)!|\mathbb{S}^n|}\int\limits_{B_1(x)} \log\frac{1}{|x-y|} f(y)\mathrm{d}y \tag{2.7}$$

where $o(1) \to 0$ as $|x| \to \infty$.

Proof. Based on the assumptions that $f \in L^\infty_{loc}(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$, it is not difficult to verify that $\mathcal{L}(f)$ is well-defined. Choose $|x| \geq e^4$ such that $|x| \geq 2\log|x|$. Following the argument in [33], we split $\mathbb{R}^n$ into three pieces

$$A_1 = B_1(x), \quad A_2 = B_{\log|x|}(0), \quad A_3 = \mathbb{R}^n \backslash (A_1 \cup A_2).$$

For $y \in A_2$ and $|y| \geq 2$, we have $|\log\frac{|x|\cdot|y|}{|x-y|}| \leq \log(2\log|x|)$. Respectively, for $|y| \leq 2$, $|\log\frac{|x|\cdot|y|}{|x-y|}| \leq |\log|y|| + C$. Thus

$$\left|\int\limits_{A_2} \log\frac{|y|}{|x-y|} f\mathrm{d}y + \log|x|\int\limits_{A_2} f\mathrm{d}y\right| \leq C\log\log|x| + C = o(1)\log|x|, \tag{2.8}$$

as $|x| \to \infty$. For $y \in A_3$, it is not hard to check

$$\frac{1}{|x|+1} \leq \frac{|y|}{|x-y|} \leq |x|+1.$$

With help of this estimate, we could control the integral over A_3 as

$$\left|\int\limits_{A_3} \log\frac{|y|}{|x-y|} f\mathrm{d}y\right| \leq \log(|x|+1)\int\limits_{A_3} |f|\mathrm{d}y. \tag{2.9}$$

For $y \in B_1(x)$, one has $1 \leq |y| \leq |x|+1$ and then

$$\left| \int_{A_1} \log|y| f \mathrm{d}y \right| \leq \log(|x|+1) \int_{A_1} |f| \mathrm{d}y.$$

Since $f \in L^1(\mathbb{R}^n)$, notice that $\int_{A_3 \cup A_1} |f| \mathrm{d}y \to 0$ as $|x| \to \infty$ and

$$\frac{2}{(n-1)!|\mathbb{S}^n|} \int_{A_2} f(y) \mathrm{d}y = \alpha + o(1).$$

Thus there holds

$$\mathcal{L}(f)(x) = (-\alpha + o(1)) \log|x| + \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{B_1(x)} \log \frac{1}{|x-y|} f(y) \mathrm{d}y. \quad \square \tag{2.10}$$

We will now utilize the lemma above to establish integral results pertaining to $\mathcal{L}(f)$ over different balls. These estimates will play a crucial role in our proofs for the main theorems. We would like to clarify beforehand: the following lemmas are under the same assumptions as stated in Lemma 2.3.

Lemma 2.4. *For any $r_0 > 0$ fixed, there holds*

$$⨍_{B_{r_0}(x)} \mathcal{L}(f)(y) \mathrm{d}y = (-\alpha + o(1)) \log|x|. \tag{2.11}$$

Proof. Fubini's theorem yields that

$$\begin{aligned}
&\left| \int_{B_{r_0}(x)} \int_{B_1(z)} \log \frac{1}{|z-y|} f(y) \mathrm{d}y \mathrm{d}z \right| \\
\leq & \int_{B_{r_0}(x)} \int_{B_{r_0+1}(x)} |\log \frac{1}{|z-y|}| \cdot |f(y)| \mathrm{d}y \mathrm{d}z \\
\leq & \int_{B_{r_0+1}(x)} |f(y)| \mathrm{d}y \int_{B_{2r_0+1}(0)} |\log|z|| \mathrm{d}z \\
\leq & C.
\end{aligned}$$

For $|x| \gg 1$, there holds for any $y \in B_{r_0}(x)$,

$$\left| \log \frac{|y|}{|x|} \right| \leq C.$$

With help of Lemma 2.3, then we have

$$⨍_{B_{r_0}(x)} \mathcal{L}(f)(y)\mathrm{d}y = (-\alpha + o(1))\log|x|. \quad \square$$

Lemma 2.5. *For any $0 < r_1 < 1$ fixed, there holds*

$$⨍_{B_{r_1|x|}(x)} \mathcal{L}(f)(y)\mathrm{d}y = (-\alpha + o(1))\log|x|. \tag{2.12}$$

Proof. By direct computation and Fubini's theorem, one has

$$\begin{aligned}
&\int_{B_{r_1|x|}(x)} \left| \int_{B_1(z)} \log\frac{1}{|z-y|} f(y)\mathrm{d}y \right| \mathrm{d}z \\
\leq &\int_{B_{r_1|x|}(x)} \int_{B_1(z)} \frac{1}{|z-y|}|f(y)|\mathrm{d}y\mathrm{d}z \\
\leq &\int_{B_{r_1|x|}(x)} \int_{B_{r_1|x|+1}(x)} \frac{1}{|z-y|}|f(y)|\mathrm{d}y\mathrm{d}z \\
\leq &\int_{B_{r_1|x|+1}(x)} |f(y)| \int_{B_{r_1|x|}(x)} \frac{1}{|z-y|}\mathrm{d}z\mathrm{d}y \\
\leq &\int_{B_{r_1|x|+1}(x)} |f(y)| \int_{B_{2r_1|x|+1}(0)} \frac{1}{|z|}\mathrm{d}z\mathrm{d}y \\
\leq &C|x|^{n-1}.
\end{aligned}$$

Thus

$$⨍_{B_{r_1|x|}(x)} \int_{B_1(z)} \log\frac{1}{|z-y|} f(y)\mathrm{d}y\mathrm{d}z = O(|x|^{-1}). \tag{2.13}$$

Meanwhile, for $y \in B_{r_1|x|}(x)$, there holds

$$\left|\log\frac{|y|}{|x|}\right| \leq \log\frac{1}{1-r_1} + \log(1+r_1) \leq C. \tag{2.14}$$

With help of these estimates (2.13), (2.14) and Lemma 2.3, we have

$$⨍_{B_{r_1|x|}(x)} \mathcal{L}(f)(y)\mathrm{d}y = (-\alpha + o(1))\log|x|. \quad \square \tag{2.15}$$

Lemma 2.6. *For any $r_2 > 0$ fixed, there holds*

$$\fint_{B_{|x|^{-r_2}}(x)} \mathcal{L}(f)(y)\mathrm{d}y = (-\alpha + o(1))\log|x|. \tag{2.16}$$

Proof. Firstly, we need modify the estimate of Lemma 2.3. By direct computation, for $|x| \gg 1$, we have

$$\left| \int\limits_{B_1(x)\setminus B_{|x|^{-r_2}}(x)} \log\frac{1}{|x-y|} f(y)\mathrm{d}y \right| \le r_2 \log|x| \int\limits_{B_1(x)} |f|\mathrm{d}y = o(1)\log|x|.$$

Using Lemma 2.3, there holds

$$\mathcal{L}(f)(x) = (-\alpha + o(1))\log|x| + \frac{2}{(n-1)!|\mathbb{S}^n|} \int\limits_{B_{|x|^{-r_2}}(x)} \log\frac{1}{|x-y|} f(y)\mathrm{d}y.$$

When $|x| \ge 2$ and $r_2 > 0$, if $z \in B_{|x|^{-r_2}}(x)$ and $y \in B_{|z|^{-r_2}}(z)$, one has

$$|y-x| \le |y-z| + |z-x| \le |z|^{-r_2} + |x|^{-r_2} \le (|x| - |x|^{-r_2})^{-r_2} + |x|^{-r_2} \le 3|x|^{-r_2}.$$

By using Fubini's theorem as before, for $|x| \gg 1$, there holds

$$\begin{aligned}
&\left| \int\limits_{B_{|x|^{-r_2}}(x)} \int\limits_{B_{|z|^{-r_2}}(z)} \log\frac{1}{|z-y|} f(y)\mathrm{d}y\mathrm{d}z \right| \\
\le& \int\limits_{B_{|x|^{-r_2}}(x)} \int\limits_{B_{3|x|^{-r_2}}(x)} \log\frac{1}{|z-y|} |f(y)|\mathrm{d}y\mathrm{d}z \\
\le& \int\limits_{B_1(x)} |f(y)|\mathrm{d}y \int\limits_{B_{4|x|^{-r_2}}(0)} \log\frac{1}{|z|}\mathrm{d}z \\
=& \Big(\int\limits_{B_1(x)} |f(y)|\mathrm{d}y \Big) \cdot |\mathbb{S}^n| (4|x|^{-r_2})^n \frac{1 - n\log(4|x|^{-r_2})}{n^2}
\end{aligned}$$

which yields that

$$\left| \frac{1}{|B_{|x|^{-r_2}}(x)|} \int\limits_{B_{|x|^{-r_2}}(x)} \int\limits_{B_{|z|^{-r_2}}(z)} \log\frac{1}{|z-y|} f(y)\mathrm{d}y\mathrm{d}z \right|$$

$$\begin{aligned}&\leq C\left(\fint_{B_1(x)}|f(y)|\mathrm{d}y\right)(nr_2\log|x|+C)\\&=o(1)\log|x|\end{aligned}$$

since $\int_{B_1(x)}|f(y)|\mathrm{d}y=o(1)$. For any $y\in B_{|x|-r_2}(x)$, it is easy to check

$$|\log\frac{|y|}{|x|}|\leq C.$$

Thus we have

$$\fint_{B_{|x|-r_2}(x)}\mathcal{L}(f)(z)\mathrm{d}z=(-\alpha+o(1))\log|x|.\quad\square \tag{2.17}$$

Lemma 2.7. *For $r_3>0$ fixed, there holds*

$$\log\left(\fint_{B_{r_3}(x)}e^{n\mathcal{L}(f)(y)}\mathrm{d}y\right)=(-n\alpha+o(1))\log|x|.$$

Proof. On one hand, with help of Jensen's inequality and Lemma 2.4, one has

$$\fint_{B_{r_3}(x)}e^{n\mathcal{L}(f)(y)}\mathrm{d}y\geq\exp\left(\fint_{B_{r_3}(x)}n\mathcal{L}(f)(y)\mathrm{d}y\right)=e^{(-n\alpha+o(1))\log|x|}. \tag{2.18}$$

Now, we are going to deal with the upper bound. For $|y|\gg1$, there holds

$$\left|\int_{B_1(y)\setminus B_{1/4}(y)}\log\frac{1}{|y-z|}f(z)\mathrm{d}z\right|\leq C\int_{B_1(y)\setminus B_{1/4}(y)}|f(z)|\mathrm{d}z.$$

Combining with Lemma 2.3, we have

$$\mathcal{L}(f)(y)=(-\alpha+o(1))\log|y|+\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{B_{1/4}(y)}\log\frac{1}{|y-z|}f(z)\mathrm{d}z.$$

Then for $|x|\gg1$ and $y\in B_{1/4}(x)$, one has

$$\mathcal{L}(f)(y)\leq(-\alpha+o(1))\log|x|+\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{B_{1/2}(x)}\log\frac{1}{|y-z|}f^+(z)\mathrm{d}z \tag{2.19}$$

where we have used the estimate $|y-z|\leq 1$ in this case.

We claim that for $|x|\gg 1$,

$$\int\limits_{B_{1/4}(x)} e^{n\mathcal{L}(f)(y)}\mathrm{d}y \leq e^{(-n\alpha+o(1))\log|x|}. \tag{2.20}$$

If $f^+(z)=0$ a.e. on $B_{1/2}(x)$, we immediately obtain (2.20) due to (2.19). Otherwise, Jensen's inequality yields that

$$\begin{aligned}
&\int\limits_{B_{1/4}(x)} e^{n\mathcal{L}(f)(y)}\mathrm{d}y\\
\leq&|x|^{-n\alpha+o(1)}\int\limits_{B_{1/4}(x)}\exp\left(\frac{2n}{(n-1)!|\mathbb{S}^n|}\int\limits_{B_{1/2}(x)}\log\frac{1}{|y-z|}f^+(z)\mathrm{d}z\right)\mathrm{d}y\\
\leq&|x|^{-n\alpha+o(1)}\int\limits_{B_{1/4}(x)}\int\limits_{B_{1/2}(x)}|y-z|^{-\frac{2n\|f^+\|_{L^1(B_{1/2}(x))}}{(n-1)!|\mathbb{S}^n|}}\frac{f^+(z)}{\|f^+\|_{L^1(B_{1/2}(x))}}\mathrm{d}z\mathrm{d}y.
\end{aligned}$$

Since $f\in L^1(\mathbb{R}^n)$, there exists $R_2>0$ such that $|x|\geq R_2$, we have

$$\|f^+\|_{L^1(B_{1/2}(x))}\leq\frac{(n-1)!|\mathbb{S}^n|}{4n}.$$

Applying Fuibini's theorem, we prove the claim. For $r_3>0$ fixed, we can choose finite balls $1\leq j\leq C(r_3)$ such that $B_{r_3}(x)\subset\cup_j B_{1/4}(x_j)$ with $x_j\in B_{r_3}(x)$. Hence, using the estimate (2.20), for $|x|\gg 1$, we have

$$\log\fint\limits_{B_{r_3}(x)} e^{n\mathcal{L}(f)(y)}\mathrm{d}y\leq(-n\alpha+o(1))\log|x|.$$

Combining with (2.18), we finish our proof. □

Lemma 2.8. *For $R\gg 1$, there holds*

$$\fint\limits_{B_{R+1}(0)\setminus B_{R-1}(0)} e^{n\mathcal{L}(f)}\mathrm{d}y=R^{-n\alpha+o(1)}$$

where $o(1)\to 0$ as $R\to\infty$.

Proof. By a direct computation, we obtain the inequality $C^{-1}R^{n-1}\leq|B_{R+1}(0)\setminus B_{R-1}(0)|\leq CR^{n-1}$ for $R\gg 1$, where C is independent of R. We can select an index

$C^{-1}R^{n-1} \leq i_R \leq CR^{n-1}$ such that the balls $B_{1/4}(x_j)$ with $|x_j| = R$ and $1 \leq j \leq i_R$ are pairwise disjoint, and the sum of the balls $B_4(x_j)$ cover the annulus $B_{R+1}(0)\backslash B_{R-1}(0)$. Applying Lemma 2.7, we obtain the following result

$$\begin{aligned}
\fint_{B_{R+1}(0)\backslash B_{R-1}(0)} e^{n\mathcal{L}(f)}\mathrm{d}y \leq & \frac{1}{C^{-1}R^{n-1}} \sum_{j=1}^{i_R} \int_{B_4(x_j)} e^{n\mathcal{L}(f)}\mathrm{d}y \\
\leq & CR^{1-n} \sum_{j=1}^{i_R} |x_j|^{-n\alpha+o(1)} \\
\leq & CR^{1-n} \cdot CR^{n-1} \cdot R^{-n\alpha+o(1)} \\
= & R^{-n\alpha+o(1)}.
\end{aligned}$$

Similarly,

$$\fint_{B_{R+1}(0)\backslash B_{R-1}(0)} e^{n\mathcal{L}(f)}\mathrm{d}y \geq \frac{1}{CR^{n-1}} \sum_{j=1}^{i_R} \int_{B_{1/4}(x_j)} e^{n\mathcal{L}(f)}\mathrm{d}y = R^{-n\alpha+o(1)}.$$

Finally, we get the desired result. □

Lemma 2.9. *There holds*

$$\mathcal{V}(e^{n\mathcal{L}(f)}) = (1-\alpha)^+.$$

Moreover, if $\alpha > 1$, one has

$$\int_{\mathbb{R}^n} e^{n\mathcal{L}(f)}\mathrm{d}x < +\infty.$$

Proof. For $R \gg 1$, choose $x_R \in \mathbb{R}^n$ with $|x_R| = \frac{R}{2}$. With help of Lemma 2.5 and Jensen's equality, we have

$$\begin{aligned}
\int_{B_R(0)} e^{n\mathcal{L}(f)}\mathrm{d}x \geq & \int_{B_{|x_R|/2}(x_R)} e^{n\mathcal{L}(f)}\mathrm{d}x \\
\geq & |B_{|x_R|/2}(x_R)| \exp\left(\fint_{B_{|x_R|/2}(x_R)} n\mathcal{L}(f)\mathrm{d}x \right) \\
\geq & CR^{n-n\alpha+o(1)}.
\end{aligned}$$

By taking $R \to \infty$, we have

$$\lim_{R\to\infty}\inf\frac{\log\int_{B_R(0)}e^{n\mathcal{L}(f)}\mathrm{d}x}{\log|B_R(0)|}\geq 1-\alpha.$$

Since for any $R\geq 1$,

$$0<\int\limits_{B_1(0)}e^{n\mathcal{L}(f)}\mathrm{d}x\leq\int\limits_{B_R(0)}e^{n\mathcal{L}(f)}\mathrm{d}x,$$

we obtain that

$$\mathcal{V}_{inf}(e^{n\mathcal{L}(f)})\geq(1-\alpha)^+. \tag{2.21}$$

Now, we claim that if $\alpha>1$, there holds

$$\int\limits_{\mathbb{R}^n}e^{n\mathcal{L}(f)}\mathrm{d}x<+\infty.$$

If $\alpha>1$, due to Lemma 2.8, there exist $R_1>1$ such that for any $R\geq R_1$, one has

$$⨍\limits_{B_{R+1}(0)\setminus B_{R-1}(0)}e^{n\mathcal{L}(f)}\mathrm{d}x\leq R^{-n\alpha+\frac{n(\alpha-1)}{2}}.$$

Then

$$\begin{aligned}\int\limits_{B_R(0)}e^{n\mathcal{L}(f)}\mathrm{d}x\leq&\int\limits_{B_{R_1}(0)}e^{n\mathcal{L}(f)}\mathrm{d}x+\sum_{i=[R_1]+1}^{[R]}\int\limits_{B_{i+1}(0)\setminus B_{i-1}(0)}e^{n\mathcal{L}(f)}\mathrm{d}x.\\ \leq&C+\sum_{i=[R_1]+1}^{[R]}Ci^{n-1}\cdot i^{-n\alpha+\frac{n(\alpha-1)}{2}}\\ =&C+C\sum_{i=[R_1]+1}^{[R]}i^{-1-\frac{n(\alpha-1)}{2}}.\end{aligned}$$

Due to $\alpha>1$, the right side is finite as $R\to\infty$. Thus we prove our claim. Immediately, for $\alpha>1$, one has

$$\mathcal{V}_{sup}(e^{n\mathcal{L}(f)})=0. \tag{2.22}$$

On the other hand, if $\alpha\leq 1$, for any small $\epsilon\in(0,1)$, with help of Lemma 2.8 again, there exists $R_\epsilon>1$ such that for any $R\geq R_\epsilon$, one has

$$⨍\limits_{B_{R+1}(0)\setminus B_{R-1}(0)}e^{n\mathcal{L}(f)}\mathrm{d}x\leq CR^{n-1-n\alpha+\epsilon}. \tag{2.23}$$

Similarly as before, there holds

$$\begin{aligned}\int\limits_{B_R(0)} e^{n\mathcal{L}(f)}\mathrm{d}x \leq & \int\limits_{B_{R_\epsilon}(0)} e^{n\mathcal{L}(f)}\mathrm{d}x + \sum_{i=[R_\epsilon]+1}^{[R]} \int\limits_{B_{i+1}(0)\setminus B_{i-1}(0)} e^{n\mathcal{L}(f)}\mathrm{d}x.\\ \leq & C(\epsilon) + C\sum_{i=[R_\epsilon]+1}^{[R]} Ci^{n-1-n\alpha+\epsilon}\\ \leq & C(\epsilon) + C\int\limits_{[R_\epsilon]}^{[R]+1} t^{n-1-n\alpha+\epsilon}\mathrm{d}t\\ \leq & C(\epsilon) + C\frac{1}{n-n\alpha+\epsilon}\left(([R]+1)^{n-n\alpha+\epsilon} - ([R_\epsilon]+1)^{n-n\alpha+\epsilon}\right)\\ \leq & C(\epsilon) + C(\epsilon)(R+1)^{n-n\alpha+\epsilon}.\end{aligned}$$

By letting $R\to\infty$, there holds

$$\mathcal{V}_{sup}(e^{n\mathcal{L}(f)}) \leq 1-\alpha+\frac{\epsilon}{n}. \tag{2.24}$$

Due to the arbitrary choice of ϵ and (2.22), we have

$$\mathcal{V}_{sup}(e^{n\mathcal{L}(f)}) \leq (1-\alpha)^+. \tag{2.25}$$

Combining (2.21) with (2.25), one has

$$\mathcal{V}(e^{n\mathcal{L}(f)}) = (1-\alpha)^+. \quad \square$$

The above lemmas provide integral results concerning the asymptotic behavior of $\mathcal{L}(f)$. If we impose additional conditions on f, we can obtain some pointwise estimates that are of independent interest similar to the two-dimensional case in [16] and [17].

Lemma 2.10. *If f^+ has compact support, there holds*

$$\mathcal{L}(f)(x) \leq -\alpha\log|x| + C, \quad |x|\gg 1.$$

Conversely, if f^- has compact support, there holds

$$\mathcal{L}(f)(x) \geq -\alpha\log|x| - C, \quad |x|\gg 1.$$

Proof. By direct computation, we have

$$\frac{(n-1)!|\mathbb{S}^n|}{2}(\mathcal{L}(f)(x) + \alpha\log|x|)$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} f \mathrm{d}y + \int_{\mathbb{R}^n} \log \frac{|y|}{|y|+1} f \mathrm{d}y \\
&= \int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} f^+ \mathrm{d}y - \int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} f^- \mathrm{d}y + C.
\end{aligned}
$$

For $|x| \geq 1$, it is easy to check that

$$
\frac{|x| \cdot (|y|+1)}{|x-y|} \geq 1
$$

which shows that

$$
\log \frac{|x| \cdot (|y|+1)}{|x-y|} \geq 0.
$$

If f^+ has compact support, there exists $R_1 > 0$ such that $supp(f^+) \subset B_{R_1}(0)$. And for $|x| \geq 2R_1$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} f^+ \mathrm{d}y \\
=& \int_{B_{R_1}(0)} \log \frac{|x| \cdot (|y|+1)}{|x-y|} f^+ \mathrm{d}y \\
\leq& \log (2|R_1|+2) \int_{B_{R_1}(0)} f^+ \mathrm{d}y \\
\leq& C
\end{aligned}
$$

which yields that

$$
\mathcal{L}(f)(x) \leq -\alpha \log |x| + C, \quad |x| \gg 1.
$$

Reversely, if f^- has compact support, following the same argument, there holds

$$
\int_{\mathbb{R}^n} \log \frac{|x| \cdot (|y|+1)}{|x-y|} f^- \mathrm{d}y \leq C, \quad |x| \gg 1
$$

and then

$$
\mathcal{L}(f)(x) \geq -\alpha \log |x| - C, \quad |x| \gg 1. \quad \square
$$

Interestingly, $\mathcal{L}(f)$ exhibits integral estimates for derivatives that are similar to the estimates for Green's representation on compact manifolds (see Lemma 2.3 in [37]). The

following lemma plays an important role in the proof of the necessity for being normal solutions.

Lemma 2.11. *For $R \gg 1$, there holds*

$$\int\limits_{B_R(0)} |\mathcal{L}(f)|\mathrm{d}x = O((\log R)\cdot R^n).$$

In particular, for $n \geq 4$, one has

$$\int\limits_{B_R(0)} |\Delta\mathcal{L}(f)|\mathrm{d}x = O(R^{n-2}), \qquad \int\limits_{B_R(0)} |\nabla\mathcal{L}(f)|^2\mathrm{d}x = O(R^{n-2}).$$

Proof. A direct computation and Fubini's theorem yield that

$$\begin{aligned}
&\int\limits_{B_R(0)} |\mathcal{L}(f)|\mathrm{d}x\\
\leq& C\int\limits_{B_R(0)}\int\limits_{\mathbb{R}^n\setminus B_{2R}(0)} |\log\frac{|y|}{|x-y|}|\cdot|f(y)|\mathrm{d}y\mathrm{d}x\\
&+C\int\limits_{B_R(0)}\int\limits_{B_{2R}(0)} |\log\frac{|y|}{|x-y|}|\cdot|f(y)|\mathrm{d}y\mathrm{d}x\\
\leq& CR^n + CR^n\int\limits_{B_1(0)} |\log|y||\cdot|f(y)|\mathrm{d}y\\
&+CR^n\int\limits_{B_{2R}(0)\setminus B_1(0)} |\log|y||\cdot|f(y)|\mathrm{d}y\\
&+\int\limits_{B_{2R}(0)} |f(y)|\int\limits_{B_{3R}(0)} |\log|z||\mathrm{d}z\\
\leq& CR^n + C(\log R)R^n\\
\leq& C(\log R)R^n.
\end{aligned}$$

Similarly, for $n \geq 4$, applying Fubini's theorem again, we have

$$\begin{aligned}
&\int\limits_{B_R(0)} |\Delta\mathcal{L}(f)|\mathrm{d}x\\
\leq& C\int\limits_{B_R(0)}\int\limits_{\mathbb{R}^n} \frac{1}{|x-y|^2}|f(y)|dy\mathrm{d}x
\end{aligned}$$

$$\begin{aligned}
\leq & C \int\limits_{B_R(0)} \int\limits_{\mathbb{R}^n \setminus B_{2R}(0)} \frac{1}{|x-y|^2} |f(y)| \mathrm{d}y \mathrm{d}x \\
& + C \int\limits_{B_R(0)} \int\limits_{B_{2R}(0)} \frac{1}{|x-y|^2} |f(y)| \mathrm{d}y \mathrm{d}x \\
\leq & C R^{n-2} + C \int\limits_{B_{2R}(0)} |f(y)| \mathrm{d}y \int\limits_{B_{3R}(0)} \frac{1}{|x|^2} \mathrm{d}x \\
\leq & C R^{n-2}.
\end{aligned}$$

Using Hölder's inequality and the assumption $f \in L^1(\mathbb{R}^n)$, there holds

$$\begin{aligned}
\int\limits_{B_R(0)} |\nabla \mathcal{L}(f)|^2 \mathrm{d}x = & C \int\limits_{B_R(0)} \left| \int\limits_{\mathbb{R}^n} \frac{x-y}{|x-y|^2} f(y) \mathrm{d}y \right|^2 \mathrm{d}x \\
\leq & C \int\limits_{B_R(0)} \left(\int\limits_{\mathbb{R}^n} \frac{1}{|x-y|^2} |f(y)| \mathrm{d}y \int\limits_{\mathbb{R}^n} |f(y)| \mathrm{d}y \right) \mathrm{d}x \\
\leq & C \int\limits_{B_R(0)} \int\limits_{\mathbb{R}^n} \frac{1}{|x-y|^2} |f(y)| \mathrm{d}y \mathrm{d}x \\
\leq & C R^{n-2}. \quad \square
\end{aligned}$$

The following Liouville-type theorem may be well-known to experts. For reader's convenience, we would like to give the proof by slightly modifying Theorem 5 in [39].

Lemma 2.12. *For any integer $m \geq 1, k \geq 0$, suppose that $(-\Delta)^m \varphi(x) = 0$ and*

$$\int\limits_{B_R(0)} \varphi^+ \mathrm{d}x = o(R^{k+n+1}).$$

Then $\varphi(x)$ is a polynomial with $\deg \varphi \leq \max\{2m-2, k\}$.

Proof. With help of Proposition 4 in [39] (or Theorem 2.2.7, p. 29 in [23]) as well as Pizzetti's formula (See [42] or Lemma 3 in [39]) for polyharmonic functions, for any $x \in \mathbb{R}^n$ and $R > |x|$, we have

$$\begin{aligned}
|D^{l+1} \varphi(x)| \leq & \frac{C}{R^{l+1}} \frac{1}{|B_R(x)|} \int\limits_{B_R(x)} |\varphi| \mathrm{d}y \\
\leq & \frac{C}{R^{l+1}} \frac{2^n}{|B_{2R}(0)|} \int\limits_{B_{2R}(0)} |\varphi| \mathrm{d}y
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C\cdot 2^n}{R^{l+1}}\frac{1}{|B_{2R}(0)|}\int\limits_{B_{2R}(0)}(-\varphi+2\varphi^+)\mathrm{d}y\\
&= \frac{C\cdot 2^n}{R^{l+1}}\left(-\sum_{i=0}^{m-1}c_i(2R)^{2i}\Delta^i\varphi(0)\right)+\frac{C\cdot 2^n}{R^{l+1}}\frac{1}{|B_{2R}(0)|}\int\limits_{B_{2R}(0)}2\varphi^+\mathrm{d}y\\
&\leq CR^{2m-2-l-1}+o(R^{k-l})
\end{aligned}
$$

where c_i are positive constants. If $k > 2m-2$, choose $l = k$ and otherwise choose $l = 2m-2$. By letting $R\to\infty$, we show that $\varphi(x)$ must be a polynomial of degree at most l. Thus we finish our proof. □

With above preparations, we will give the proof of Theorem 2.2.

Proof of Theorem 2.2. Set $P(x) := w(x) - \mathcal{L}(f)$. By using Green's function (2.3), we obtain

$$(-\Delta)^{\frac{n}{2}}P(x) = 0. \tag{2.26}$$

Firstly, we show that $P(x)$ is a polynomial function. Since $w(x)$ exhibits polynomial volume growth, we can find $s \geq 0$ such that

$$\int\limits_{B_R(0)} e^{nw}\mathrm{d}x = O(R^s). \tag{2.27}$$

Let $q\in(0,1)$ be a small number that satisfies $(\frac{s}{n}-1)q \leq \frac{1}{2}$. Then by using Hölder's inequality and Lemma 2.11, we have

$$
\begin{aligned}
\int\limits_{B_R(0)} P^+\mathrm{d}x &\leq \int\limits_{B_R(0)} w^+\mathrm{d}x + \int\limits_{B_R(0)}|\mathcal{L}(f)|\mathrm{d}x\\
&\leq \int\limits_{B_R(0)}\frac{1}{q}e^{qw}\mathrm{d}x + C(\log R)R^n\\
&\leq \frac{1}{q}\Big(\int\limits_{B_R(0)} e^{nu}\mathrm{d}x\Big)^{\frac{q}{n}}\Big(\int\limits_{B_R(0)}\mathrm{d}x\Big)^{1-\frac{q}{n}} + C(\log R)R^n\\
&\leq CR^{\frac{sq}{n}+n-q} + C(\log R)R^n\\
&\leq CR^{n+\frac{1}{2}}
\end{aligned}
$$

where we have used the fact $qw^+ \leq e^{qw}$. Lemma 2.12 concludes that $P(x)$ is a polynomial function with a degree no greater than $n-2$.

When $n = 2$, we immediately have $P(x) \equiv C$. As for the case $n \geq 4$, we will show that $P(x)$ has a finite upper bound. Since $P(x)$ is a polynomial of degree at most $n-2$, there holds $|\nabla P(x)| \leq C|x|^{n-3}$ for $|x| \gg 1$ where C is independent of x. Then for $y \in B_{R^{3-n}}(x)$, one has

$$P(y) \geq P(x) - C.$$

With help of this estimate, there holds

$$\begin{aligned}
\int\limits_{B_{2R}(0)} e^{nw}\mathrm{d}y \geq & \int\limits_{B_{R^{3-n}}(x)} e^{nw}\mathrm{d}y \\
\geq & Ce^{nP(x)} \int\limits_{B_{R^{3-n}}(x)} e^{n\mathcal{L}(f)}\mathrm{d}y \\
\geq & Ce^{nP(x)}|B_{R^{3-n}}(x)| \exp\left(\frac{1}{|B_{R^{3-n}}(x)|}\int\limits_{B_{R^{3-n}}(x)} n\mathcal{L}(f)\mathrm{d}y\right) \\
\geq & Ce^{nP(x)}R^C
\end{aligned}$$

where we make use of Lemma 2.6 in the last estimate. Based our assumption on the polynomial volume growth on w, then one has

$$P(x) \leq C\log(|x|+2). \tag{2.28}$$

Finally, we discuss the necessary and sufficient conditions for $P(x)$ to be a constant for $n \geq 4$. On one hand, we consider the case (a). If $P(x) \equiv C$, by Lemma 2.11, we have

$$\int\limits_{B_R(0)} |\Delta w|\mathrm{d}x = \int\limits_{B_R(0)} |\Delta\mathcal{L}(f)|\mathrm{d}x = O(R^{n-2}) = o(R^n).$$

Conversely, if (2.5) holds, due to Lemma 2.11, there holds

$$\int\limits_{B_R(0)} |\Delta P|\mathrm{d}x \leq \int\limits_{B_R(0)} \left(|\Delta w| + |\Delta\mathcal{L}(f)|\right)\mathrm{d}x = o(R^n). \tag{2.29}$$

We have $\Delta P = 0$ since ΔP is also a polynomial. Due to the upper bound (2.28), Liouville's theorem yields that $P(x)$ must be a constant.

On the other hand, we deal with the case (b). When $P(x)$ is a constant, due to Lemma 2.11, there holds

$$\int\limits_{B_R(0)} |w|\mathrm{d}x \leq \int\limits_{B_R(0)} (|\mathcal{L}(f)| + C)\mathrm{d}x = O(\log R \cdot R^n) + O(R^n) = o(R^{n+2}).$$

Conversely, if (2.6) holds, with help of Lemma 2.11 again, one has

$$\int\limits_{B_R(0)} |P(x)|\mathrm{d}x \leq \int\limits_{B_R(0)} (|w| + |\mathcal{L}(f)|)\mathrm{d}x = o(R^{n+2}).$$

Then, due to $P(x)$ is polynomial and the upper bound (2.28), we must have $P(x) \equiv C$.

Finally, we finish our proof. □

In fact, with help of a Liouville type theorem Lemma 2.12, we will get a more general decomposition theorem as follows.

Theorem 2.13. *Let $n \geq 2$ be an even integer and consider a smooth solution $w(x)$ to (2.2) with smooth $f \in L^1(\mathbb{R}^n)$. Suppose that for some integer $k \geq 0$,*

$$⨍\limits_{B_R(0)} w^+\mathrm{d}x = o(R^{k+1}). \tag{2.30}$$

Then

$$w(x) = \mathcal{L}(f) + P(x)$$

where $P(x)$ is a polynomial with $\deg(P) \leq \max\{k, n-2\}$.

Proof. Set $P(x) = w(x) - \mathcal{L}(f)$ and we have

$$(-\Delta)^{\frac{n}{2}} P(x) = 0.$$

With help of Lemma 2.11 and the assumption (2.30), there holds

$$\begin{aligned} ⨍\limits_{B_R(0)} P^+\mathrm{d}x \leq & ⨍\limits_{B_R(0)} w^+\mathrm{d}x + ⨍\limits_{B_R(0)} |\mathcal{L}(f)|\mathrm{d}x \\ \leq & o(R^{k+1}) + C\log R \\ = & o(R^{k+1}). \end{aligned}$$

Applying Lemma 2.12, one has $P(x)$ is a polynomial and $\deg(P) \leq \max\{k, n-2\}$. □

With help of the decomposition in Theorem 2.2, we can establish the following volume comparison theorem.

Theorem 2.14. *Consider a smooth solution $w(x)$ to (2.2), where $n \geq 2$ is an even integer with smooth $f \in L^1(\mathbb{R}^n)$.*

(a) *Assume that* $\mathcal{V}_{sup}(e^{nw}) < +\infty$. *Then we have* $\mathcal{V}_{\sup}(e^{nw}) \leq (1-\alpha)^+$. *If* $\alpha > 1$ *in addition, the volume is finite i.e.* $e^{nw} \in L^1(\mathbb{R}^n)$.
(b) *If* $w(x)$ *is a normal solution, then the limit of volume growth exists and*

$$\mathcal{V}(e^{nw}) = (1-\alpha)^+. \tag{2.31}$$

Proof. If $\mathcal{V}_{sup}(e^{nw}) < +\infty$, with help of Theorem 2.2, we have the decomposition

$$w(x) = \mathcal{L}(f)(x) + P(x) \tag{2.32}$$

where $P(x)$ is a polynomial and $P(x) \leq C$. Then

$$\int\limits_{B_R(0)} e^{nw}\mathrm{d}x = \int\limits_{B_R(0)} e^{n\mathcal{L}(f)(x)+nP(x)}\mathrm{d}x \leq C \int\limits_{B_R(0)} e^{n\mathcal{L}(f)}\mathrm{d}x \tag{2.33}$$

which yields that

$$\mathcal{V}_{sup}(e^{nw}) \leq \mathcal{V}_{sup}(e^{n\mathcal{L}(f)}). \tag{2.34}$$

Applying Lemma 2.9 and the estimate (2.34), one has

$$\mathcal{V}_{sup}(e^{nw}) \leq (1-\alpha)^+.$$

If $\alpha > 1$, Lemma 2.9 combining with (2.33) deduces that

$$\int\limits_{\mathbb{R}^n} e^{nw}\mathrm{d}x < +\infty.$$

Specially, if $w(x)$ is a normal solution, there holds

$$w(x) = \mathcal{L}(f)(x) + C.$$

Due to Lemma 2.9, then the limit of volume growth exists and

$$\mathcal{V}(e^{nw}) = (1-\alpha)^+.$$

Thus we finish our proof. □

Remark 2.15. It is easy to see from (2.31) that the volume is infinite for normal solutions with $\alpha < 1$. However, the case $\alpha = 1$ is more delicate. It is possible for both infinite and finite volume to occur. See Section 6.

Until now, we haven't considered the assumption on the completeness of the metric. It is also interesting to note that if we replace the completeness assumption with the assumption of finite volume, we obtain a reversed Cohn-Vossen inequality. For reader's convenience, for $n = 2$, we rewrite (1.3) as

$$-\Delta u = Ke^{2u} \tag{2.35}$$

where both K and u are smooth functions on $\mathbb{R}^2$.

Theorem 2.16. *Consider the equation (2.35). If the volume is finite and the negative part of Gaussian curvature is integrable i.e.*

$$\int_{\mathbb{R}^2} e^{2u}\mathrm{d}x < +\infty \quad \text{and} \quad \int_{\mathbb{R}^2} K^- e^{2u}\mathrm{d}x < +\infty,$$

then there holds

$$\int_{\mathbb{R}^2} Ke^{2u}\mathrm{d}x \geq 2\pi. \tag{2.36}$$

Proof. If the integral $\int_{\mathbb{R}^2} K^+ e^{2u}\mathrm{d}x = \infty$, the inequality (2.36) is automatically satisfied due to the condition $\int_{\mathbb{R}^2} K^- e^{2u}\mathrm{d}x < \infty$. Therefore, we only need to consider the case where $\int_{\mathbb{R}^2} K^+ e^{2u}\mathrm{d}x < \infty$, which implies that the total Gaussian curvature is finite. Under the assumption of finite volume, the volume entropy $\tau(g)$ is zero. According to Theorem 2.2, the solution u must be a normal solution. By utilizing Theorem 2.14, we obtain

$$\int_{\mathbb{R}^2} Ke^{2u}\mathrm{d}x \geq 2\pi. \quad \square$$

Remark 2.17. Actually, for the two-dimensional case, Lytchak provided a geometric proof (Theorem 1.6 in [35]) and as he expected in Remark 4.1 of [35], we present an analytical proof based on the works of Huber [27] here.

For higher-order cases, the finite volume assumption alone is insufficient to establish a lower bound on the total Q-curvature (refer to [10]). In order to obtain a similar result to Theorem 2.16, additional restrictions on Δu are required to ensure that the solution is normal, utilizing Theorem 2.2.

Theorem 2.18. *Consider the equation (1.3) with $n \geq 4$ is an even integer. Suppose that*

$$\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x < +\infty, \quad \int_{\mathbb{R}^n} Q_g^- e^{nu}\mathrm{d}x < +\infty$$

and

$$\int\limits_{B_R(0)} |\Delta u| \mathrm{d}x = o(R^n). \tag{2.37}$$

Then there holds

$$\int\limits_{\mathbb{R}^n} Q_g e^{nu} \mathrm{d}x \geq \frac{(n-1)!|\mathbb{S}^n|}{2}.$$

Proof. The proof is essentially the same as that of Theorem 2.16, and therefore we omit it here. □

3. Strong A_∞ weight and length comparison

Consider the measure distance

$$\delta(x,y) = \left(\int\limits_{B_{x,y}} e^{nu} \mathrm{d}z \right)^{\frac{1}{n}}$$

where $B_{x,y}$ is the Euclidean ball in $\mathbb{R}^n$ with diameter $|x-y|$ that contains x and y i.e.

$$B_{x,y} = B_{\frac{|x-y|}{2}}(\frac{x+y}{2}).$$

For a positive locally integrable function $\varphi(x)$ on $\mathbb{R}^n$, we say $\varphi(x)$ is an A_∞ weight if for all balls B there holds

$$\frac{1}{|B|} \int\limits_B \varphi \mathrm{d}x \leq C \exp\left(\frac{1}{|B|} \int\limits_B \log \varphi \mathrm{d}x \right). \tag{3.1}$$

More equivalent definitions of A_∞ weight can be found in [43]. The definition of A_∞ weight (3.1) taken here was firstly introduced by Hruščev in [28]. If we suppose the volume term e^{nu} is an A_∞ weight, an important property of A_∞ weight (See Proposition 3.12 in [43] or Lemma 3.5 in [1]) shows that

$$d_g(x,p) \leq C\delta(x,p) \tag{3.2}$$

where C is a constant independent of x and p. In [21], David and Semmes introduced a strong A_∞ weight which is an A_∞ weight additionally satisfying

$$d_g(x,p) \geq C\delta(x,p). \tag{3.3}$$

Briefly, for a strong A_∞ weight, the measure distance and geodesic distance are equivalent. In harmonic analysis, A_p weight has many interesting properties and we refer the interested reader to Chapter V of Stein's well-known monograph [44]. With help of the property of strong A_∞ weight, we can generalize a result of Li and Tam (See Corollary 3.3 in [31]).

Lemma 3.1. *Suppose that $g = e^{2u}|dx|^2$ is a normal metric on $\mathbb{R}^n$ where $n \geq 2$ is an even integer and $u(x)$ satisfies (1.3) with $Q_g e^{nu} \in L^1(\mathbb{R}^n)$, for fixed $p \in \mathbb{R}^n$, there holds*

$$\lim_{R\to\infty} \frac{\log V_g(B_R(p))}{\log |B_R(p)|} = \lim_{|x|\to\infty} \frac{\log \delta(x,p)}{\log |x-p|} = (1-\alpha_0)^+.$$

Proof. Since $u(x)$ is a normal solution, we have the decomposition using logarithmic potential:

$$u(x) = \mathcal{L}(Q_g e^{nu})(x) + C.$$

With help of Theorem 2.14, there holds

$$\lim_{R\to\infty} \frac{\log V_g(B_R(0))}{\log |B_R(0)|} = (1-\alpha_0)^+. \tag{3.4}$$

For each fixed $p \in \mathbb{R}^n$ and $R \gg 1$, we have

$$\int\limits_{B_{R/2}(0)} e^{nu}\mathrm{d}x \leq \int\limits_{B_R(p)} e^{nu}\mathrm{d}x \leq \int\limits_{B_{2R}(0)} e^{nu}\mathrm{d}x$$

which yields that

$$\lim_{R\to\infty} \frac{\log V_g(B_R(p))}{\log |B_R(p)|} = (1-\alpha_0)^+.$$

For $|x| \gg 1$, it is not hard to check that

$$B_{\frac{1}{8}|x|}(\frac{x}{2}) \subset B_{x,p} = B_{\frac{|x-p|}{2}}(\frac{x+p}{2}).$$

Then, with help of Jensen's inequality and Lemma 2.5, it follows that

$$\begin{aligned}\log \delta(x,p) \geq & \frac{1}{n}\log \int\limits_{B_{\frac{1}{8}|x|}(\frac{x}{2})} e^{nu}\mathrm{d}y \\ = & \frac{1}{n}\log \fint\limits_{B_{\frac{1}{8}|x|}(\frac{x}{2})} e^{nu}\mathrm{d}y + \log|x| + \log\frac{|\mathbb{S}^n|^{\frac{1}{n}}}{8}\end{aligned}$$

$$\geq \fint_{B_{\frac{1}{8}|x|}(\frac{x}{2})} u\mathrm{d}y + \log|x| + C$$
$$\geq (1-\alpha_0+o(1))\log|x| + C$$

which yields that

$$\liminf_{|x|\to\infty} \frac{\log\delta(x,p)}{\log|x-p|} \geq 1-\alpha_0.$$

It is not hard to check $|B_1(p)\cap B_{x,p}| \geq \frac{|\mathbb{S}^n|}{4}$ for $|x| \gg 1$ and then

$$\liminf_{|x|\to\infty} \frac{\log\delta(x,p)}{\log|x-p|} \geq 0$$

which yields that

$$\liminf_{|x|\to\infty} \frac{\log\delta(x,p)}{\log|x-p|} \geq (1-\alpha_0)^+. \tag{3.5}$$

For $|x| \gg 1$, one may easily check that

$$\delta(x,p)^n \leq \fint_{B_{2|x|}(0)} e^{nu}\mathrm{d}y.$$

Apply the estimate (3.4) to get

$$\limsup_{|x|\to\infty} \frac{\log\delta(x,p)}{\log|x-p|} \leq (1-\alpha_0)^+.$$

Combining with (3.5), one has

$$\lim_{|x|\to\infty} \frac{\log\delta(x,p)}{\log|x-p|} = (1-\alpha_0)^+. \quad \square$$

Now, we are going to give the proof of Theorem 1.4.

Proof of Theorem 1.4. Choose the curve $\gamma(t) = t\frac{x}{|x|}$ and the point $p_1 = |p|\frac{x}{|x|}$. Then one has

$$d_g(x,p) \leq d_g(p_1,p) + d_g(x,p_1) \leq C + \int_{|p|}^{|x|} e^{u(\gamma(t))}\mathrm{d}t.$$

To estimate the second term, we firstly claim that for $|x| \gg 1$, there holds

$$\int\limits_{|x|-\frac{1}{2}}^{|x|+\frac{1}{2}} e^{u(\gamma(t))}\mathrm{d}t \leq |x|^{-\alpha_0+o(1)}. \tag{3.6}$$

Using Lemma 2.3 and the assumption $u(z)$ is normal, we have

$$u(z) \leq (-\alpha_0+o(1))\log|z| + \frac{2}{(n-1)!|\mathbb{S}^n|}\int\limits_{B_1(z)} \log\frac{1}{|z-y|}Q_g^+(y)e^{nu(y)}\mathrm{d}y. \tag{3.7}$$

By slightly modifying Lemma 2.3, we can easily get

$$u(z) \leq (-\alpha_0+o(1))\log|z| + \frac{2}{(n-1)!|\mathbb{S}^n|}\int\limits_{B_{1/4}(z)} \log\frac{1}{|z-y|}Q_g^+(y)e^{nu(y)}\mathrm{d}y.$$

If $Q_g^+ = 0$ a.e. on $B_1(x)$, for any $z \in B_{1/2}(x)$, there holds

$$u(z) \leq (-\alpha_0+o(1))\log|x|.$$

Then the claim follows in this case. Otherwise, since $Q_g e^{nu} \in L^1(\mathbb{R}^n)$, there exists $R_1 > 0$ such that for any $|x| > R_1$,

$$\int\limits_{B_1(x)} Q_g^+(y)e^{nu(y)}\mathrm{d}y \leq \frac{(n-1)!|\mathbb{S}^n|}{4}.$$

The estimate (3.7) and Jensen's inequality yield that for $|x| > R_1$ and $z \in B_{1/2}(x)$

$$\begin{aligned}
e^{u(z)} =& Ce^{\mathcal{L}(Q_g e^{nu})}\\
\leq& |x|^{-\alpha_0+o(1)}\exp\left(\frac{2}{(n-1)!|\mathbb{S}^n|}\int\limits_{B_1(x)}\log\frac{1}{|z-y|}Q_g^+(y)e^{nu}\mathrm{d}y\right)\\
\leq& |x|^{-\alpha_0+o(1)}\int\limits_{B_1(x)}\left(\frac{1}{|z-y|}\right)^{\frac{2\|Q_g^+e^{nu}\|_{L^1(B_1(x))}}{(n-1)!|\mathbb{S}^n|}}\frac{Q_g^+(y)e^{nu}}{\|Q_g^+e^{nu}\|_{L^1(B_1(x))}}\mathrm{d}y\\
\leq& |x|^{-\alpha_0+o(1)}\int\limits_{B_1(x)}\frac{1}{\sqrt{|z-y|}}\frac{Q_g^+(y)e^{nu}}{\|Q_g^+e^{nu}\|_{L^1(B_1(x))}}\mathrm{d}y.
\end{aligned}$$

Then by using Fubini's theorem and the above estimate, we have

$$\int\limits_{|x|-\frac{1}{2}}^{|x|+\frac{1}{2}} e^{u(\gamma(t))}\mathrm{d}t \leq |x|^{-\alpha_0+o(1)}\int\limits_{|x|-\frac{1}{2}}^{|x|+\frac{1}{2}}\int\limits_{B_1(x)}\frac{1}{\sqrt{|\gamma(t)-y|}}\frac{Q_g^+(y)e^{nu(y)}}{\|Q_g^+e^{nu}\|_{L^1(B_1(x))}}\mathrm{d}y\mathrm{d}t$$

$$\begin{aligned}&\leq C|x|^{-\alpha_0+o(1)}\\&=|x|^{-\alpha_0+o(1)}\end{aligned}$$

Thus we prove our claim (3.6).

For any $\epsilon > 0$, the estimate (3.6) shows that there exist $R_2 > 0$ such that for $|x| \geq R_2$

$$\int_{|x|-\frac{1}{2}}^{|x|+\frac{1}{2}} e^{u(\gamma(t))}\mathrm{d}t \leq |x|^{-\alpha_0+\epsilon}.$$

Then we have

$$d_g(x,p) \leq C + \int_{|p|}^{|x|} e^{u(\gamma(t))}\mathrm{d}t \leq C(\epsilon) + \sum_{i=[R_2]+2}^{[|x|]+1} i^{-\alpha_0+\epsilon}. \tag{3.8}$$

If $\alpha_0 > 1$, by choosing small ϵ and using (3.8), one has

$$\lim_{|x|\to\infty} \sup d_g(x,p) < +\infty. \tag{3.9}$$

Moreover, for any $x, y \in \mathbb{R}^n$, considering a fixed point p, notice that $d_g(x,y) \leq d_g(x,p) + d_g(p,y)$ to get

$$\mathrm{diam}(\mathbb{R}^n, e^{2u}|dx|^2) < +\infty.$$

Since, for $|x| \gg 1$, any curves connecting the points x and p must cross the sphere $\partial B_1(p)$, then

$$d_g(x,p) \geq \inf_{|y-p|=1} d_g(y,p) > 0$$

which shows that

$$\lim_{|x|\to\infty} \inf \frac{\log d_g(x,p)}{\log|x-p|} \geq 0. \tag{3.10}$$

Obviously, combining (3.9) with (3.10), one has

$$\lim_{|x|\to\infty} \frac{\log d_g(x,p)}{\log|x-p|} = 0.$$

Now, we will deal with the case $\alpha_0 \leq 1$. From (3.8), due to the monotonicity of $s^{-\alpha_0+\epsilon}$, one has

$$d_g(x,p) \le C(\epsilon) + \int_{[R_2]+1}^{[|x|]+2} s^{-\alpha_0+\epsilon} \mathrm{d}s \le C(\epsilon) + \frac{([|x|]+2)^{1-\alpha_0+\epsilon}}{1-\alpha_0+\epsilon}$$

which yields that

$$\lim_{|x|\to\infty} \sup \frac{\log d_g(x,p)}{\log|x-p|} \le 1-\alpha_0+\epsilon.$$

By the arbitrary choice of ϵ, we obtain

$$\lim_{|x|\to\infty} \sup \frac{\log d_g(x,p)}{\log|x-p|} \le 1-\alpha_0. \tag{3.11}$$

Thus for the case $\alpha_0 = 1$, combining with (3.10), one has

$$\lim_{|x|\to\infty} \frac{\log d_g(x,p)}{\log|x-p|} = 0.$$

If $\alpha_0 < 1$, it was shown in Corollary 1.7 of [46] that e^{nu} is a strong A_∞ weight which deduces that

$$C^{-1}\delta(x,p) \le d_g(x,p) \le C\delta(x,p). \tag{3.12}$$

Using Lemma 3.1, one has

$$\lim_{|x|\to\infty} \frac{\log d_g(x,p)}{\log|x-p|} = 1-\alpha_0.$$

Finally, we finish our proof. □

Theorem 3.2. *Suppose that $g = e^{2u}|dx|^2$ is a complete and normal metric. There holds*

$$\lim_{R\to\infty} \frac{\log V_g(B_R(p))}{\log|B_R(p)|} = \lim_{|x|\to\infty} \frac{\log d_g(x,p)}{\log|x-p|} = 1-\alpha_0. \tag{3.13}$$

Proof. In fact, since g is a complete and normal metric, Theorem 1.3 in [11] (See also [40], [24]) has shown that $\alpha_0 \le 1$. In fact, applying Theorem 1.4, it is not hard to show that $\alpha_0 \le 1$ because if g is complete, for fixed $p \in \mathbb{R}^n$, we have $d_g(x,p) \to \infty$ as $|x| \to \infty$. Combining Lemma 3.1 with Theorem 1.4, we finish our proof. □

4. Necessary and sufficient conditions for normal solutions

Now, return to the conformal invariant equation (1.3) with finite total Q-curvature. As we defined in Section 2, we say $u(x)$ is a normal solution to (1.3) if $u(x) = \mathcal{L}(Q_g e^{nu}) + C$

where C is a constant. Therefore, we need to outline a criterion to identify normal solutions. In [11], Chang, Qing and Yang demonstrated that solutions are normal when $R_g \geq 0$ near infinity. Similarly, Wang et al. showed in [45] that solutions are normal if $\int_{\mathbb{R}^n} (R_g^-)^{\frac{n}{2}} e^{nu} \mathrm{d}x < +\infty$. In the following theorem, we want to generalize their results. Our proof is based on similar principles as theirs, with emphasizing the necessity of being normal solutions, a previously overlooked consideration.

Before stating our theorem, we establish a lemma which will be used in the proof.

Lemma 4.1. *If $h : \mathbb{R}^n \to \mathbb{R}$ satisfies $\Delta^m h = 0$ for some integer $m > 2$ and for $x \in \mathbb{R}^n$:*

$$\fint_{B_R(x)} \Delta h \mathrm{d}y + c \fint_{B_R(x)} |\nabla h|^2 \mathrm{d}y \leq o(1) \tag{4.1}$$

where $c > 0$ and $o(1) \to 0$ as $R \to \infty$. Then one has

$$\Delta^{m-1} h = \Delta^{m-2} dh = 0.$$

Proof. Using the condition (4.1), we immediately have

$$\int_{B_R(x)} \Delta h \mathrm{d}y \leq o(R^n). \tag{4.2}$$

With help of Pizzetti's formula (See [42] or Lemma 3 in [39]) for polyharmonic functions, we have

$$\frac{1}{|B_R(x)|} \int_{B_R(x)} \Delta h \mathrm{d}y = \sum_{i=0}^{m-2} c_i R^{2i} \Delta^i (\Delta h(x))$$

where c_i are positive constants. By letting $R \to \infty$, the estimate (4.2) yields that the leading term

$$\Delta^{m-1} h(x) \leq 0.$$

With help of Liouville's theorem for the harmonic function $\Delta^{m-1} h$, we have

$$\Delta^{m-1} h = C_0 \leq 0 \tag{4.3}$$

for some constant C_0. Consequently, for any $1 \leq j \leq n$, one has

$$\Delta^{m-1} \partial_j h = 0.$$

Applying Pizzetti's formula to $\partial_j h$, there holds

$$\frac{1}{|B_R(x)|}\int\limits_{B_R(x)} \partial_j h \mathrm{d}y = \sum_{i=0}^{m-2} c_i R^{2i} \Delta^i \partial_j h(x).$$

Hölder's inequality and the estimate (4.8) yield that

$$\begin{aligned}
&\frac{1}{|B_R(x)|}\int\limits_{B_R(x)} \Delta h \mathrm{d}y + c(\frac{1}{|B_R(x)|}\int\limits_{B_R(x)} \partial_j h \mathrm{d}y)^2 \\
\leq &\frac{1}{|B_R(x)|}\int\limits_{B_R(x)} \Delta h \mathrm{d}y + c\frac{1}{|B_R(x)|}\int\limits_{B_R(x)} |\nabla h|^2 \mathrm{d}y \\
\leq &o(1)
\end{aligned}$$

where $o(1) \to 0$ as $R \to \infty$. Since $m > 2$, the leading term of the left side is

$$(c_{m-2}(\Delta^{m-2}\partial_j h(x))^2 R^{4m-8}$$

which concludes that

$$\Delta^{m-2}\partial_j h = 0.$$

Immediately, one has

$$\Delta^{m-1} h = 0. \quad \square$$

Theorem 4.2. *Consider a smooth function $u(x)$ that satisfies equation (1.3) with even integer $n \geq 4$ and $Q_g e^{nu} \in L^1(\mathbb{R}^n)$. Then $u(x)$ is a normal solution if and only if the following condition is satisfied:*

$$\int\limits_{B_R(0)} R_g^- e^{2u} \mathrm{d}x = o(R^n). \tag{4.4}$$

Proof. For brevity, set $v(x) = \mathcal{L}(Q_g e^{nu})$. With help of Green's function (2.3), one has

$$(-\Delta)^{n/2}(u - v) = 0.$$

Set $h := u - v$ which is a polyharmonic function satisfying

$$(-\Delta)^{\frac{n}{2}} h = 0. \tag{4.5}$$

Consider $R > 2|x|$ and then Lemma 2.11 yields that

$$\int\limits_{B_R(x)} |\Delta v(z)|\mathrm{d}z \leq \int\limits_{B_{2R}(0)} |\Delta v(z)|\mathrm{d}z \leq CR^{n-2} \tag{4.6}$$

as well as

$$\int\limits_{B_R(x)} |\nabla v|^2\mathrm{d}z \leq \int\limits_{B_{2R}(0)} |\nabla v|^2\mathrm{d}z \leq CR^{n-2}. \tag{4.7}$$

On one hand, if (4.4) holds, with help of the definition R_g (1.4), for $R > |x|$, one has

$$\int\limits_{B_R(x)} \Delta u\mathrm{d}y + \frac{n-2}{2} \int\limits_{B_R(x)} |\nabla u|^2\mathrm{d}y \leq \frac{1}{2(n-1)} \int\limits_{B_{2R}(0)} R_g^- e^{2u}\mathrm{d}y = o(R^n).$$

Based on the estimate $|\nabla h|^2 \leq 2|\nabla u|^2 + 2|\nabla v|^2$ and the estimates (4.6), (4.7), there holds

$$\int\limits_{B_R(x)} \Delta h\mathrm{d}y + \frac{n-2}{4} \int\limits_{B_R(x)} |\nabla h|^2\mathrm{d}y \leq o(R^n). \tag{4.8}$$

Using Lemma 4.1, we have

$$\Delta^{\frac{n}{2}-1}h = \cdots = \Delta^2 h = 0;\ \Delta^{\frac{n}{2}-1}\partial_j h = \cdots = \Delta\partial_j h = 0.$$

With help of (4.8), one has

$$⨍\limits_{B_R(x)} \Delta h\mathrm{d}y \leq o(1).$$

With help of Liouville theorem for harmonic function Δh, there holds

$$\Delta h = C_1 \tag{4.9}$$

where C_1 is a non-positive constant. Then, insert the estimate (4.9) into (4.8) to get

$$\frac{n-2}{4}\frac{1}{|B_R(x)|} \int\limits_{B_R(x)} |\nabla h|^2\mathrm{d}y \leq o(1) - \frac{1}{|B_R(x)|} \int\limits_{B_R(x)} \Delta h\mathrm{d}y = -C_1 + o(1).$$

By mean value property for harmonic functions and letting $R \to \infty$, there holds

$$\partial_j h(x) = \frac{1}{|B_R(x)|} \int\limits_{B_R(x)} \partial_j h\mathrm{d}y$$

$$\leq (\frac{1}{|B_R(x)|}\int\limits_{B_R(x)} |\nabla h|^2 \mathrm{d}y)^{\frac{1}{2}}$$

$$\leq C.$$

Liouville's theorem yields that $\partial_j h$ are constants and then $\Delta h = 0$. Then using (4.8) again as well as the result $\partial_j h$ are constants, one has

$$\partial_j h = 0,\ 1 \leq j \leq n$$

which concludes that h is a constant. Finally, the solution u is normal. Reversely, if u is a normal solution, it is not hard to check that (4.4) holds due to Lemma 2.11.

Thus $u(x)$ is a normal solution if and only if (4.4) holds. □

Remark 4.3. Regrettably, the condition outlined in (4.4) appears to lack a clear geometric interpretation. Nevertheless, one can utilize (4.4) alongside Hölder's inequality to establish that $\int_{B_R(0)} (R_g^-)^{\frac{n}{2}} e^{nu} \mathrm{d}x = o(R^n)$ is sufficient for the identification of normal solutions. However, it remains challenging to demonstrate whether this condition is also necessary.

We will now give the proof of Theorem 1.1 which provides a different perspective to confirm the normal solution compared with Theorem 4.2.

Proof of Theorem 1.1. When the metric is complete and normal, Theorem 3.2 shows that $\tau(g)$ is finite and

$$\tau(g) = 1 - \frac{2}{(n-1)!|\mathbb{S}^n|}\int\limits_{\mathbb{R}^n} Q_g e^{nu} \mathrm{d}x.$$

Conversely, if $\tau(g)$ is finite, with help of Theorem 2.2, one has the decomposition

$$u(x) = \mathcal{L}(Q_g e^{nu}) + P(x)$$

where $P(x)$ is a polynomial of degree at most $n-2$ and $P(x) \leq C\log(|x|+2)$. When $n = 2$, $P(x)$ is already a constant. When $n \geq 4$, we will show that $P(x)$ must be a constant based on the completeness assumption of the metric. We argue by contradiction. If $P(x)$ is not a constant, since $P(x) \leq C\log(|x|+2)$ and $\deg(P) \leq n-2$, there exists $k \geq 1$ such that $P(x) = H_{2k}(x) + P_{2k-1}(x)$ where $H_{2k}(x)$ is a non-positive homogeneous polynomial and $P_{2k-1}(x)$ is a polynomial of degree at most $2k-1$. Then there exists $x_0 \in \mathbb{R}^n$ with $|x_0| = 1$ and $t_0 > 0$ such that on the ray $\gamma(t) = tx_0$ there holds $P(\gamma(t)) \leq -Ct^{2k}$ for $t \geq t_0 > 0$. With help of the estimate (3.6), there exists $\epsilon_1 > 0$ and $t_2 > 0$ such that for any $t \geq t_2$ there holds

$$\int_t^{t+1} e^{\mathcal{L}(Q_g e^{nu})(\gamma(s))} \mathrm{d}s \leq t^{-\alpha_0+\epsilon_1}.$$

Then choosing a integer $T_1 = [t_0 + t_1 + 1]$ and integer $T > T_1$, one has

$$\begin{aligned} d_g(\gamma(T_1), \gamma(T)) &\leq \int_{T_1}^{T} e^{u(\gamma(t))} \mathrm{d}t \\ &\leq \sum_{i=T_1}^{T} \int_i^{i+1} e^{u(\gamma(t))} \mathrm{d}t \\ &\leq \sum_{i=T_1}^{T} e^{-Ci^{2k}} \int_i^{i+1} e^{\mathcal{L}(Q_g e^{nu})(\gamma(t))} \mathrm{d}t \\ &\leq \sum_{i=T_1}^{T} e^{-Ci^{2k}} i^{-\alpha_0+\epsilon_1} < +\infty \end{aligned}$$

which contradicts to $d_g(\gamma(T_1), \gamma(T)) \to \infty$ as $T \to \infty$. Thus $P(x)$ must be a constant i.e. the metric is normal.

Finally, we finish our proof. □

Proof of Corollary 1.3. With help of the finite volume entropy assumption and Theorem 1.1, we show that the metric is normal and

$$\tau(g) = 1 - \frac{2}{(n-1)!|\mathbb{S}^n|} \int_{\mathbb{R}^n} Q_g e^{nu} \mathrm{d}x.$$

Since $Q_g \geq 0$, it is obvious that $\tau(g) \leq 1$. If $\tau(g) = 1$, it is easy to see $Q_g \equiv 0$ and then $u \equiv C$ due to $u(x)$ is normal. Conversely, when $u \equiv C$, it is trivial to show that $\tau(g) = 1$. □

5. Polynomial growth polyharmonic functions

In this section, we will give the proof of Theorem 1.7 related to polynomial growth polyharmonic functions with help of Theorem 3.2.

Proof of Theorem 1.7. Case(i):
When the volume entropy is finite, Theorem 1.1 shows that the metric is normal. With help of Theorem 3.2, for fixed $p \in \mathbb{R}^n$ and any $\epsilon > 0$, there exists $R(\epsilon) > 0$ such that for any $|x| \geq R(\epsilon)$, there holds

$$d_g(x, p) \leq |x - p|^{\tau(g)+\epsilon}. \tag{5.1}$$

If $f(x) \in \mathcal{PH}_d(M,g)$, the above estimate yields that

$$|f(x)| \leq Cd_g(x,p)^d + C \leq C|x-p|^{d(\tau(g)+\epsilon)} + C$$

which yields that

$$|f(x)| \leq C|x|^{d(\tau(g)+\epsilon)} + C$$

It is not hard to check that $d(\tau(g)+\epsilon) < [d(\tau(g)+\epsilon)] + 1$. Then we have

$$\int\limits_{B_R(0)} |f|\mathrm{d}x = o(R^{n+[d(\tau(g)+\epsilon)]+1}).$$

Due to the conformal invariant of GJMS operator, $P_g f(x) = 0$ is equivalent to

$$(-\Delta)^{\frac{n}{2}} f(x) = 0.$$

Then with help of Lemma 2.12, $f(x)$ must be a polynomial with $\deg(f) \leq \max\{n-2, [d(\tau(g)+\epsilon)]\}$. Since the polynomial $|f(x)| \leq C|x|^{d(\tau(g)+\epsilon)} + C$, we further obtain

$$\deg(f) \leq [d(\tau(g)+\epsilon)]. \tag{5.2}$$

Then we have

$$\dim(\mathcal{PH}_d(M,g)) \leq \dim(\mathcal{PH}_{[d(\tau(g)+\epsilon)]}(\mathbb{R}^n, |dx|^2)). \tag{5.3}$$

For sufficiently small ϵ, one may check

$$[d(\tau(g)+\epsilon)] \leq d\tau(g).$$

Thus we show that $f(x) \in \mathcal{PH}_{d\tau(g)}(\mathbb{R}^n, |dx|^2)$ which yields that

$$\dim(\mathcal{PH}_d(M,g)) \leq \dim(\mathcal{PH}_{d\tau(g)}(\mathbb{R}^n, |dx|^2)). \tag{5.4}$$

Case(ii):
When $\tau(g) = 0$, with help of (5.4), we have

$$\dim(\mathcal{PH}_d(M,g)) \leq \dim(\mathcal{PH}_0(\mathbb{R}^n, |dx|^2)) = 1.$$

Obviously, since constant functions belongs to the kernel of P_g, we have

$$\dim(\mathcal{PH}_d(M,g)) \geq 1.$$

Thus the identity (1.10) holds for $\tau(g) = 0$.

When $Q_g \geq 0$ near infinity, with help of Lemma 2.10, for $|x| \gg 1$, there holds

$$u(x) \geq -\alpha_0 \log |x| - C. \tag{5.5}$$

As the argument in Lemma 3.1, for $|x| \gg 1$, one has $B_{\frac{1}{8}|x|}(\frac{x}{2}) \subset B_{x,p}$ and then using (5.5) to get

$$\begin{aligned}
\delta(x,p)^n \geq & \int\limits_{B_{\frac{1}{8}|x|}(\frac{x}{2})} e^{nu(y)} \mathrm{d}y \\
\geq & C \int\limits_{B_{\frac{1}{8}|x|}(\frac{x}{2})} |y|^{-n\alpha_0} \mathrm{d}y \\
\geq & C|x|^{n-n\alpha_0} \\
= & C|x|^{n\tau(g)}.
\end{aligned}$$

Then, for $\tau(g) > 0$ and the estimate (3.12), we have

$$d_g(x,p) \geq C|x|^{\tau(g)}. \tag{5.6}$$

For each $f \in \mathcal{PH}_{d\tau(g)}(\mathbb{R}^n, |dx|^2)$, the estimate (5.6) deduces that

$$|f(x)| \leq C|x|^{d\tau(g)} + C \leq Cd(x,p)^d + C$$

which shows that f also belongs to $\mathcal{PH}_d(M,g)$. Thus we have

$$\dim(\mathcal{PH}_d(M,g)) \geq \dim(\mathcal{PH}_{d\tau(g)}(\mathbb{R}^n, |dx|^2)).$$

Combining with (5.4), we obtain the identity (1.10).

Case(iii):

When $Q_g \geq 0$, it is easy to get $\tau(g) \leq 1$ due to (1.9). With help of (5.4) and $\tau(g) \leq 1$, it follows that for each integer $k \geq 1$

$$\dim(\mathcal{PH}_k(M,g)) \leq \dim(\mathcal{PH}_k(\mathbb{R}^n, |dx|^2)).$$

If equality holds, we claim that $\tau(g) = 1$. We argue by contradiction. If $\tau(g) < 1$, due to the estimate (5.2), we can choose ϵ to be sufficiently small such that $\tau(g) + \epsilon < 1$ and then $[k(\tau(g)+\epsilon)] < k$ for the integer $k \geq 1$. Thus

$$\dim(\mathcal{PH}_k(M,g)) \leq \dim(\mathcal{PH}_{[k(\tau(g)+\epsilon)]}(\mathbb{R}^n, |dx|^2)) < \dim(\mathcal{PH}_k(\mathbb{R}^n, |dx|^2))$$

which is a contradiction. When $\tau(g) = 1$, we can deduce that $Q_g \equiv 0$ from (1.9) since $Q_g \geq 0$. Consequently, the normal solution $u(x)$ must be a constant. Conversely, if $u(x)$ is constant, the equality obviously holds.

Case(iv):

When we have $\tau(g) = 0$, for each $d \geq 0$ and each $f \in \mathcal{PH}_d(M,g)$, the estimate (5.2) shows that f must be a constant by choosing small ϵ. Conversely, if $\mathcal{PH}_d(M,g)$ consists solely of constant functions for each $d \geq 0$, we claim that $\tau(g) = 0$. We proceed by contraposition, assuming that $\tau(g) > 0$. By using Theorem 3.2 to obtain the inequality $|x-p|^{\tau(g)-\epsilon_1} \leq d_g(x,p)$ for $|x| \gg 1$, where $\epsilon_1 = \frac{\tau(g)}{2}$. Then, we have $|x| \leq C + C d_g(x,p)^{\frac{2}{\tau(g)}}$. Since $P_g x_1 = 0$, we obtain $x_1 \in \mathcal{PH}_{\frac{2}{\tau(g)}}(M,g)$, which leads to a contradiction.

Finally, we finish our proof. □

6. Examples

Now, we are going to construct some examples to show the case $\alpha_0 = 1$ is very subtle by following the examples introduced in [27]. Choose a smooth cut-off function $\eta(x)$ satisfying $\eta(x) = 1$ in $B_{10}(0)$ and vanishes on $\mathbb{R}^n \backslash B_{20}(0)$. Consider the function

$$w(x) = (1-\eta(x))\,(-\log|x| + c\log\log|x|)$$

where c is a constant. It is not hard to see $w(x)$ is a smooth function on $\mathbb{R}^n$. For $|x| \geq 30$, by using polar coordinates $|x| = r$, one can check that for $|x| \geq 30$, there holds

$$|(-\Delta)^{\frac{n}{2}} w(x)| \leq \frac{C}{r^n (\log r)^2}$$

which shows that

$$\int_{\mathbb{R}^n} |(-\Delta)^{\frac{n}{2}} w| \mathrm{d}x < +\infty.$$

Consider the metric $g = e^{2w}|dx|^2$ and set the Q-curvature as

$$Q_g = e^{-nw}(-\Delta)^{\frac{n}{2}} w.$$

It is not hard to check that

$$\int_{B_R(0)} e^{nw} \mathrm{d}x = O(R), \qquad \int_{B_R(0)} |w| \mathrm{d}x = o(R^{n+1}).$$

By using Theorem 2.2, we find that the metric g is normal. Applying Lemma 2.4, we must have

$$\int_{\mathbb{R}^n} Q_g e^{nw} \mathrm{d}x = \frac{(n-1)!|\mathbb{S}^n|}{2}.$$

Choosing $x_0 \in \mathbb{R}^n$ with $|x_0| = 1$, for $R > 30$, direct computation yields that

$$\int_{30}^{R} e^{u(tx_0)} \mathrm{d}t = \int_{30}^{R} t^{-1} (\log t)^c \mathrm{d}t, \qquad \int_{B_R(0)\setminus B_{30}(0)} e^{nw} \mathrm{d}x = |\mathbb{S}^n| \int_{30}^{R} t^{-1} (\log t)^{nc} \mathrm{d}t.$$

If $c < -1$, the diameter of $(\mathbb{R}^n, e^{2w}|dx|^2)$ must be finite. For $c \geq -1$, one can check that the metric is complete. Similarly, when $c < -\frac{1}{n}$, the volume is finite, whereas for $c \geq -\frac{1}{n}$, the volume is infinite.

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References

[1] C.L. Aldana, G. Carron, S. Tapie, A_∞-weights and compactness of conformal metrics under $L^{n/2}$ curvature bounds, Anal. PDE 14 (7) (2021) 2163–2205.
[2] N. Aronszajn, T.M. Creese, L.J. Lipkin, Polyharmonic Functions. Notes Taken by Eberhard Gerlach, Oxford Mathematical Monographs. Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, ISBN 0-19-853906-1, 1983, x+265 pp.
[3] P. Aviles, Conformal complete metrics with prescribed nonnegative Gaussian curvature in $\mathbb{R}^2$, Invent. Math. 83 (3) (1986) 519–544.
[4] S. Axler, P. Bourdon, W. Ramey, Harmonic Function Theory, second edition, Graduate Texts in Mathematics, vol. 137, Springer-Verlag, New York, ISBN 0-387-95218-7, 2001, xii+259 pp.
[5] T.P. Branson, Differential operators canonically associated to a conformal structure, Math. Scand. 57 (1985) 2.
[6] S. Brendle, Global existence and convergence for a higher order flow in conformal geometry, Ann. Math. (2) 158 (1) (2003) 323–343.
[7] M. Bonk, J. Heinonen, E. Saksman, Logarithmic potentials, quasiconformal flows, and Q-curvature, Duke Math. J. 142 (2) (2008) 197–239.
[8] G. Carron, Euclidean volume growth for complete Riemannian manifolds, Milan J. Math. 88 (2) (2020) 455–478.
[9] G. Carron, M. Herzlich, The Huber theorem for non-compact conformally flat manifolds, Comment. Math. Helv. 77 (2002) 192–220.
[10] S.-Y.A. Chang, W. Chen, A note on a class of higher order conformally covariant equations, Discrete Contin. Dyn. Syst. 7 (2) (2001) 275–281.
[11] S.-Y.A. Chang, J. Qing, P.C. Yang, On the Chern-Gauss-Bonnet integral for conformal metrics on $\mathbb{R}^4$, Duke Math. J. 103 (3) (2000) 523–544.
[12] S.-Y.A. Chang, J. Qing, P.C. Yang, Compactification of a class of conformally flat 4-manifold, Invent. Math. 142 (2000) 65–93.
[13] S.-Y.A. Chang, P.C. Yang, Extremal metrics of zeta function determinants on 4-manifolds, Ann. Math. (2) 142 (1) (1995) 171–212.
[14] J. Cheeger, T.H. Colding, W.P. Minicozzi II, Linear growth harmonic functions on complete manifolds with nonnegative Ricci curvature, Geom. Funct. Anal. 5 (6) (1995) 948–954.

[15] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (1991) 615–622.
[16] W. Chen, C. Li, Qualitative properties of solutions to some nonlinear elliptic equations in $\mathbf{R}^2$, Duke Math. J. 71 (2) (1993) 427–439.
[17] K.-S. Cheng, C.-S. Lin, On the asymptotic behavior of solutions of the conformal Gaussian curvature equations in $\mathbf{R}^2$, Math. Ann. 308 (1) (1997) 119–139.
[18] S. Cohn-Vossen, Kürzeste Wege und Totalkrümmung auf Flächen, Compos. Math. 2 (1935) 69–133.
[19] T.H. Colding, W.P. Minicozzi II, Harmonic functions on manifolds, Ann. Math. (2) 146 (3) (1997) 725–747.
[20] T.H. Colding, W.P. Minicozzi II, Harmonic functions with polynomial growth, J. Differ. Geom. 46 (1) (1997) 1–77.
[21] G. David, S. Semmes, Strong A_∞ weights, Sobolev inequalities and quasiconformal mappings, in: Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 122, Marcel Dekker, 1990, pp. 101–111.
[22] Z. Djadli, A. Malchiodi, Existence of conformal metrics with constant Q-curvature, Ann. Math. (2) 168 (3) (2008) 813–858.
[23] L.C. Evans, Partial Differential Equations, Grad. Stud. in Math., vol. 19, Amer. Math. Soc., 1998.
[24] H. Fang, On a conformal Gauss–Bonnet–Chern inequality for LCF manifolds and related topics, Calc. Var. 23 (2005) 469–496.
[25] R. Finn, On a class of conformal metrics, with application to differential geometry in the large, Comment. Math. Helv. 40 (1965) 1–30.
[26] C. Graham, R. Jenne, L. Mason, G. Sparling, Conformally invariant powers of the Laplacian. I: existence, J. Lond. Math. Soc., II. Ser. 46 (3) (1992) 557–565.
[27] A. Huber, On subharmonic functions and differential geometry in the large, Comment. Math. Helv. 32 (1957) 13–72.
[28] S. Hruščev, A description of weights satisfying the A_∞ condition of Muckenhoupt, Proc. Am. Math. Soc. 90 (1984) 253–257.
[29] A. Hyder, Conformally Euclidean metrics on $\mathbb{R}^n$ with arbitrary total Q-curvature, Anal. PDE 10 (3) (2017) 635–652.
[30] F. Ledrappier, X. Wang, An integral formula for the volume entropy with applications to rigidity, J. Differ. Geom. 85 (3) (2010) 461–477.
[31] P. Li, L. Tam, Complete surfaces with finite total curvature, J. Differ. Geom. 33 (1) (1991) 139–168.
[32] P. Li, S.-T. Yau, Linear growth harmonic functions on a complete manifold, J. Differ. Geom. 29 (1989) 421–425.
[33] M. Li, A Liouville-type theorem in conformally invariant equations, https://doi.org/10.1007/s00208-023-02712-9.
[34] C. Lin, A classification of solutions of a conformally invariant fourth order equation in R^n, Comment. Math. Helv. 73 (2) (1998) 206–231.
[35] A. Lytchak, On conformal planes of finite area, Math. Z. 305 (3) (2023) 50.
[36] S. Ma, J. Qing, On Huber-type theorems in general dimensions, Adv. Math. 395 (2022) 108145.
[37] A. Malchiodi, Compactness of solutions to some geometric fourth-order equations, J. Reine Angew. Math. 594 (2006) 137–174.
[38] A. Manning, Topological entropy for geodesic flows, Ann. Math. (2) 110 (3) (1979) 567–573.
[39] L. Martinazzi, Classification of solutions to the higher order Liouville's equation on $\mathbb{R}^{2m}$, Math. Z. 263 (2) (2009) 307–329.
[40] C. Ndiaye, J. Xiao, An upper bound of the total Q-curvature and its isoperimetric deficit for higher-dimensional conformal Euclidean metrics, Calc. Var. Partial Differ. Equ. 38 (2010) 1–27.
[41] S.A. Paneitz, A quartic conformally covariant differential operator for arbitrary pseudo-Riemannian manifolds, SIGMA 4 (2008) 036.
[42] P. Pizzetti, Sulla media dei valori che una funzione dei punti dello spazio assume alla superficie di una sfera, Rend. Lincei 18 (1909) 182–185.
[43] S. Semmes, Bi-Lipschitz mappings and strong A_∞ weights, Ann. Acad. Sci. Fenn., Ser. A 1 Math. 18 (2) (1993) 211–248.
[44] E. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality, and Oscillatory Integrals, Princeton University Press, Princeton, NJ, 1993.
[45] S. Wang, Y. Wang, Integrability of scalar curvature and normal metric on conformally flat manifolds, J. Differ. Equ. 265 (4) (2018) 1353–1370.
[46] Y. Wang, The isoperimetric inequality and quasiconformal maps on manifolds with finite total Q-curvature, Int. Math. Res. Not. 2 (2012) 394–422.

[47] Y. Wang, The isoperimetric inequality and Q-curvature, Adv. Math. 281 (2015) 823–844.
[48] J. Wei, X. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 313 (2) (1999) 207–228.
[49] J. Wei, D. Ye, Nonradial solutions for a conformally invariant fourth order equation in $\mathbb{R}^4$, Calc. Var. Partial Differ. Equ. 32 (3) (2008) 373–386.
[50] J. Xiao, Toward best isoperimetric constants for (H^1, BMO)-normal conformal metrics on $\mathbb{R}^n$, $n \geq 3$, Adv. Math. 220 (2) (2009) 540–559.
[51] X. Xu, Uniqueness and non-existence theorems for conformally invariant equations, J. Funct. Anal. 222 (1) (2005) 1–28.
[52] S.T. Yau, Harmonic functions on complete Riemannian manifolds, Commun. Pure Appl. Math. 28 (1975) 201–228.
[53] S. Zhang, The sharp type Chern-Gauss-Bonnet integral and asymptotic behavior, arXiv:2205.10817.