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A Liouville-type theorem in conformally invariant equations

Mingxiang Li[1]

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Abstract

Given a smooth function $K(x)$ satisfying a polynomially cone condition and $x \cdot \nabla K \leq 0$, we prove that there is no solution $u \in C^\infty(\mathbb{R}^2)$ of the equation

$$-\Delta u = K(x)e^{2u} \quad \text{on } \mathbb{R}^2$$

with $u \leq C$ and $\int_{\mathbb{R}^2} |K(x)|e^{2u}\mathrm{d}x < +\infty$. As a consequence, there is no such solution if $K(x)$ is a non-constant polynomial with $x \cdot \nabla K \leq 0$. The latter result already includes a result of Struwe(JEMS 2020) as a particular case. Higher order cases are set up with additional assumption on the behavior of Δu near infinity.

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1 Introduction

Recently, Borer, Galimberti and Struwe [3] studied the asymptotic behavior of the metrics of a family of sign-changing prescribed Gaussian curvature on (M, g) with $\chi(M) < 0$ and then Galimberti [11] studied it with $\chi(M) = 0$. Their blow-up analysis led possibly to either solutions of the standard Liouville equation $-\Delta u = e^{2u}$ in $\mathbb{R}^2$ or solutions to

$$-\Delta u = (1 + (Ax, x))e^{2u} \quad \text{in } \mathbb{R}^2$$

with $\int_{\mathbb{R}^2}(1 + |A(x, x)|)e^{2u}\mathrm{d}x < \infty$ where A is a negative definite 2×2-matrix. Later, Struwe [26] ruled out the possibility of the later case. In [26], Struwe established an elegant result as follows.

✉ Mingxiang Li
limx@smail.nju.edu.cn

1 Department of Mathematics, Nanjing University, Nanjing 210093, People's Republic of China

Theorem 1.1 (Theorem 1.3 in [26]) *Suppose A is a negative definite and symmetric 2×2-matrix. Then there is no solution $u \in C^\infty(\mathbb{R}^2)$ of the equation*

$$-\Delta u = (1 + (Ax, x))e^{2u} \quad \text{on } \mathbb{R}^2$$

with $u \leq C$ and such that the induced metric $h = e^{2u} g_{\mathbb{R}^2}$ has finite volume and integrated curvature

$$\int_{\mathbb{R}^2} e^{2u} \mathrm{d}x < \infty, \quad \int_{\mathbb{R}^2} (1 + (Ax, x))e^{2u} \mathrm{d}x \in \mathbb{R}.$$

Inspired by the above theorem, we establish our main theorem in current paper.

Theorem 1.2 *Suppose $p(x)$ is a non-constant polynomial with $x \cdot \nabla p \leq 0$. Then there is no solution $u \in C^\infty(\mathbb{R}^2)$ of the equation*

$$-\Delta u = p(x)e^{2u} \quad \text{on } \mathbb{R}^2$$

with $u \leq C$ and such that the induced metric $h = e^{2u} g_{\mathbb{R}^2}$ has finite total curvature

$$\int_{\mathbb{R}^2} |p(x)|e^{2u} \mathrm{d}x < \infty.$$

Remark 1.3 In Theorem 1.1, Struwe assumed that the induced metric has the finite volume and finite integrated curvature. Here, we assume the finite total curvature. Actually, in some special cases including the case in Theorem 1.1, they are equivalent (see Lemma A.2).

In conformal geometry, the following equation

$$-\Delta u = K(x)e^{2u} \quad \text{in } \mathbb{R}^2 \tag{1.1}$$

plays a very important role. In particular, when $K(x)$ is a positive constant, the classification theorem in [6] gives a very nice description of the solutions with finite volume. When $K(x) \leq 0$ and $K(x) \geq -C/|x|^l$ with $l > 2$, Ni [22] showed that the Eq. (1.1) possesses infinitely many solutions. Later, Cheng and Ni [10] showed that the Eq. (1.1) has no solution when $K(x) \leq 0$ and $K(x) \leq -C|x|^{-2}$ near infinity. Not long after that, Chen and Li [7] considered $K(x)$ positive near infinity and gave a nice description of the asymptotic behavior of the solutions to (1.1) as well as a generalized Pohozaev's identity similar to Lemma 2.1. The property that $K(x)$ is positive near infinity is crucial in their proof. In [9], Cheng and Lin gave a more detailed discussion about the asymptotic behavior of the solutions including both $K(x)$ is positive and negative at infinity with finite total curvature. McOwen [21] and Aviles [2] investigated the situation where $K(x) \to 0$ in some order as $|x| \to \infty$. If we restrict the equation on bounded domain, there are also many well-known results including [4, 15, 17] and many others. Via stereographic projection, we can pull back the metric $e^{2u} g_{\mathbb{R}^2}$ to the sphere $\mathbb{S}^2$ and then it becomes a well-known Nirenberg problem. Many distinguished works are devoted to it such as [5, 25, 30] and many others.

As for higher order cases, Galimberti [12] and Ngô-Zhang [23] studied the related problem about Q-curvature in dimension four. Recently, Struwe [27] established a fourth order theorem like Theorem 1.1. In current paper, we will generalize it to all higher order cases. Consider the higher order equation

$$(-\Delta)^{n/2}u = K(x)e^{nu} \quad \text{on } \mathbb{R}^n$$

with even $n \geq 4$. Similar to the two-dimensional case, there are also famous classification theorems when K is a positive constant, see [19, 28] and so on.

Actually, we will prove a more general theorem compared with Theorem 1.2. Now, we define a polynomially cone condition(denoted as (**PCC**) for brevity) as follows: For a given continuous function $K(x)$ with $x \in \mathbb{R}^n$, there exist $R_1 > 0, \theta > 0, \gamma \geq n$ and $c_1 > 0, c_2 > 0$ such that

$$|K(x)| \leq c_2|x|^\gamma$$

for $|x| \geq R_1$ and

$$|K(x)| \geq c_1|x|^\gamma \quad \text{in } C_\theta \backslash B_{R_1}(0)$$

where C_θ is a cone with angle θ and vertex at the origin.

When $p(x)$ is a non-constant polynomial in $\mathbb{R}^2$ with $x \cdot \nabla p \leq 0$, it is not hard to see $\deg(p(x)) \geq 2$ and then $p(x)$ satisfies (**PCC**) with help of Lemma 2.6. Therefore, Theorem 1.2 is a corollary of Theorem 1.4. Now, we state the general theorem as follows.

Theorem 1.4 *Suppose $K(x)$ is a smooth function on $\mathbb{R}^2$ satisfying* (**PCC**) *and $x \cdot \nabla K \leq 0$. Then there is no solution $u \in C^\infty(\mathbb{R}^2)$ of the equation*

$$-\Delta u = K(x)e^{2u} \quad \text{on } \mathbb{R}^2 \tag{1.2}$$

with $u \leq C$ and such that the induced metric $h = e^{2u}g_{\mathbb{R}^2}$ has finite total curvature

$$\int_{\mathbb{R}^2} |K(x)|e^{2u}\mathrm{d}x < \infty.$$

For higher order cases, we can also get similar result with additional assumption on the behavior of Δu near infinity like that in [27]. Precisely, we state it as follows.

Theorem 1.5 *Suppose $K(x)$ is a smooth function on $\mathbb{R}^n$ satisfying* (**PCC**) *and $x \cdot \nabla K \leq 0$ where $n \geq 4$ is an even integer. Then there is no solution $u \in C^\infty(\mathbb{R}^n)$ of the equation*

$$(-\Delta)^{n/2}u = K(x)e^{nu} \quad \text{on } \mathbb{R}^n \tag{1.3}$$

with $u \leq C$ and

$$\Delta u(x) \to 0 \quad \text{as } |x| \to \infty \tag{1.4}$$

such that the induced metric $h = e^{2u} g_{\mathbb{R}^n}$ has finite total curvature

$$\int_{\mathbb{R}^n} |K(x)| e^{nu} \mathrm{d}x < \infty.$$

Remark 1.6 In [27], Struwe considered $K(x) = 1 + A(x, x, x, x)$ with A being a negative definite and symmetric 4-linear map on $\mathbb{R}^4$. Based on the definition of A in Struwe's work, for any $x \in \mathbb{R}^4$, we have

$$c_2|x|^4 \geq |A(x, x, x, x)| \geq c_1|x|^4$$

for some $c_1 > 0$, $c_2 > 0$. Using this property, one has

$$|1 + A(x, x, x, x)| \geq c_1|x|^4 - 1 \geq \frac{c_1}{2}|x|^4$$

for some $R_1 > 0$ and any $|x| \geq R_1 > 0$ which shows that $1 + A(x, x, x, x)$ satisfies (**PCC**). Therefore Theorem 1 in [27] is a special case of Theorem 1.5.

Since this paper is short, we need not mention the structure of the paper as it goes naturally.

2 Crucial lemmas

The following lemma plays an important role in the proof of our main theorems. The following generalized Pohozaev identity is also got in [9] and [7] with additional assumptions but the proof in [29] is very different from others. Inspired by [29], we give the following Pohozaev identity. Moreover, with help of the ideas from [16] and [8] related to intergral equations and elliptic theory, we generalize Theorem 1.1 in [29] with lower regularity assumption of the solution.

Lemma 2.1 *Given a function $K(x) \in C^1(\mathbb{R}^n)$ where $n \geq 2$ is an even integer satisfying* (**PCC**) *and $x \cdot \nabla K \leq 0$. Suppose $w(x) \in L^\infty_{loc}(\mathbb{R}^n)$ bounded from above is a solution of the integral equation*

$$w(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^n} \left(\log \frac{|y|}{|x - y|} \right) K(y) e^{nw(y)} \mathrm{d}y \tag{2.1}$$

such that $K(x)e^{nw(x)} \in L^1(\mathbb{R}^n)$ where ω_n is the volume of unit sphere in $\mathbb{R}^{n+1}$. Then

$$\frac{4}{n\omega_n} \int_{\mathbb{R}^n} \langle x, \nabla K(x) \rangle e^{nw(x)} \mathrm{d}x = \alpha(\alpha - 2) \tag{2.2}$$

where $\alpha = \frac{2}{\omega_n} \int_{\mathbb{R}^n} K(y) e^{nw(y)} \mathrm{d}y$.

Proof Firstly, we will show that $w \in C^1(\mathbb{R}^n)$. Here, we just assume that $w \in L^\infty_{loc}(\mathbb{R}^n)$. Actually, with help of Lemma A.1, we can deduce that $w \in C^1(\mathbb{R}^n)$ when $Ke^{nw} \in L^1(\mathbb{R}^n)$, $K(x) \in C^1(\mathbb{R}^n)$ and $w \in L^\infty_{loc}(\mathbb{R}^n)$. By direct computation, one has

$$\langle x, \nabla w\rangle = -\frac{2}{\omega_n}\int_{\mathbb{R}^n}\frac{\langle x, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y \tag{2.3}$$

Multiplying by $K(x)e^{nw(x)}$ and integrating over the ball $B_R(0)$ for any $R > 0$, we have

$$\begin{aligned}\int_{B_R(0)} K(x)e^{nw(x)}&\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n}\frac{\langle x, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &= \int_{B_R(0)} K(x)e^{nw(x)}\langle x, \nabla w(x)\rangle \mathrm{d}x.\end{aligned} \tag{2.4}$$

With $x = \frac{1}{2}((x+y)+(x-y))$, for the left-hand side of (2.4), one has the following identity

$$\begin{aligned}LHS =&\frac{1}{2}\int_{B_R(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &+\frac{1}{2}\int_{B_R(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n}\frac{\langle x+y, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x.\end{aligned}$$

Now, we deal with the last term of above equation by changing variables x and y.

$$\begin{aligned}&\int_{B_R(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n}\frac{\langle x+y, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &=\int_{B_R(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n\backslash B_R(0)}\frac{\langle x+y, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &=\int_{B_{R/2}(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n\backslash B_R(0)}\frac{\langle x+y, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &\quad+\int_{B_R(0)\backslash B_{R/2}(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{\mathbb{R}^n\backslash B_{2R}(0)}\frac{\langle x+y, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &\quad+\int_{B_R(0)\backslash B_{R/2}(0)} K(x)e^{nw(x)}\left[-\frac{2}{\omega_n}\int_{B_{2R(0)}\backslash B_R(0)}\frac{\langle x+y, x-y\rangle}{|x-y|^2}K(y)e^{nw(y)}\mathrm{d}y\right]\mathrm{d}x\\ &=: I_1+I_2+I_3.\end{aligned}$$

Notice that

$$\left|I_1\right| \le 3\frac{2}{\omega_n}\int_{B_{R/2}(0)}|K(x)|e^{nw(x)}\mathrm{d}x\int_{\mathbb{R}^n\backslash B_R(0)}|K(y)\Big|e^{nw(y)}\mathrm{d}y$$

and

$$|I_2| \leq 3\frac{2}{\omega_n}\int_{B_R(0)\backslash B_{R/2}(0)}|K(x)|e^{nw(x)}\mathrm{d}x\int_{\mathbb{R}^n\backslash B_{2R}(0)}|K(y)|e^{nw(y)}\mathrm{d}y.$$

Then both $|I_1|$ and $|I_2|$ tend to zero as $R \to \infty$ due to $Ke^{nw} \in L^1(\mathbb{R}^n)$.

Now, we focus on the term I_3. With help of Lemma 2.2, for each $x \in B_R(0)\backslash B_{R/2}(0)$ and K satisfying (**PCC**), one has $|K|e^{nw(x)} \leq C|x|^{-n}$ and then

$$\begin{aligned}
&\int_{B_{2R(0)}\backslash B_R(0)}\frac{|x+y|}{|x-y|}|K(y)|e^{nw(y)}\mathrm{d}y \\
&\leq CR^{1-n}\int_{B_{2R(0)}\backslash B_R(0)}\frac{1}{|x-y|}\mathrm{d}y \\
&\leq CR^{1-n}\int_{B_{3R}(0)}\frac{1}{|y|}\mathrm{d}y \leq C.
\end{aligned}$$

Here and thereafter, we denote by C a constant which may be different from line to line.

Thus

$$|I_3| \leq C\int_{B_R(0)\backslash B_{R/2}(0)}|K|e^{nw}\mathrm{d}x \to 0$$

as $R \to \infty$.

Thus we have

$$LHS \to -\frac{1}{2}\alpha\int_{\mathbb{R}^n}K(y)e^{nw(y)}\mathrm{d}y.$$

As for the right-hand side of (2.4), by using divergence theorem, we have

$$\begin{aligned}
RHS =&\frac{1}{n}\int_{B_R(0)}K(x)\langle x, \nabla e^{nw(x)}\rangle\mathrm{d}x \\
=&-\int_{B_R(0)}\left(K(x)+\frac{1}{n}\langle x, \nabla K(x)\rangle\right)e^{nw(x)}\mathrm{d}x \\
&+\frac{1}{n}\int_{\partial B_R(0)}K(x)e^{nw(x)}R\mathrm{d}\sigma.
\end{aligned}$$

Since $K(x)e^{nw(x)} \in L^1(\mathbb{R}^n)$, the last term of above equation tends to zero as $R \to \infty$ suitably. Precisely, there exist a sequence $R_i \to \infty$ such that

$$\lim_{i\to\infty}R_i\int_{\partial B_{R_i}(0)}Ke^{nw}\mathrm{d}\sigma = 0.$$

Then

$$\lim_{i\to\infty}\frac{4}{n\omega_n}\int_{B_{R_i}(0)} x\cdot\nabla K e^{nw}\mathrm{d}x=\alpha(\alpha-2).$$

Due to $x\cdot\nabla K\le 0$, we have

$$\frac{4}{n\omega_n}\int_{\mathbb{R}^n} x\cdot\nabla K e^{nw}\mathrm{d}x=\alpha(\alpha-2).$$

□

Inspired by the proof of Theorem 4.5 in [26], we give the estimate of α as follows.

Lemma 2.2 *Under the same assumptions as that in Lemma 2.1, then we have*

$$\alpha\ge 1+\frac{\gamma}{n}\ge 2$$

and

$$\lim_{|x|\to\infty}\frac{w(x)}{\log|x|}=-\alpha.$$

Moreover,

$$w(x)\le-\alpha\log|x|+C$$

for $|x|>>1$.

Proof Choose $|x|\ge e^4$ such that $|x|\ge 2\log|x|$. We split $\mathbb{R}^n$ into three pieces

$$A_1=B_1(x),\quad A_2=B_{\log|x|}(0),\quad A_3=\mathbb{R}^n\backslash(A_1\cup A_2).$$

For $y\in A_2$ and $|y|\ge 2$, we have $|\log\frac{|x|\cdot|y|}{|x-y|}|\le\log(2\log|x|)$. Respectively, for $|y|\le 2$, $|\log\frac{|x|\cdot|y|}{|x-y|}|\le|\log|y||+C$. Thus

$$\left|\int_{A_2}\log\frac{|y|}{|x-y|}Ke^{nw}\mathrm{d}y+\log|x|\int_{A_2}Ke^{nw}\mathrm{d}y\right|\le C\log\log|x|+C=o(1)\log|x|. \tag{2.5}$$

as $|x|\to\infty$. For $y\in A_3$, it is not hard to check

$$\frac{1}{|x|+1}\le\frac{|y|}{|x-y|}\le|x|+1.$$

With help of this estimate, we control the integral over A_3 as

$$\left|\int_{A_3}\log\frac{|y|}{|x-y|}Ke^{nw}\mathrm{d}y\right|\le\log(|x|+1)\int_{A_3}|K|e^{nw(y)}\mathrm{d}y. \tag{2.6}$$

For $y \in B_1(x)$, one has $1 \le |y| \le |x| + 1$ and then

$$\left| \int_{A_1} \log |y| K e^{nw} \mathrm{d}y \right| \le \log(|x| + 1) \int_{A_1} |K| e^{nw} \mathrm{d}y.$$

It is not hard to see $\int_{A_3 \cup A_1} |K| e^{nw(y)} \mathrm{d}y \to 0$ as $|x| \to \infty$. Thus we have

$$\begin{aligned} &\frac{2}{\omega_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} K(y) e^{nw(y)} \mathrm{d}y \\ &\quad = (-\alpha + o(1)) \log |x| + \frac{2}{\omega_n} \int_{A_1} \log \frac{1}{|x-y|} K(y) e^{nw(y)} \mathrm{d}y. \end{aligned} \tag{2.7}$$

Due to $K e^{nw} \in L^1(\mathbb{R}^n)$ and Fubini's theorem, one has

$$\int_{B_1(x_0)} \left| \int_{A_1} \log \frac{1}{|x-y|} K(y) e^{nw(y)} \mathrm{d}y \right| \mathrm{d}x \le C.$$

Thus

$$\int_{B_1(x_0)} \left| \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} K e^{nw} \mathrm{d}y \right| \mathrm{d}x \le C \log(|x_0| + 2) \tag{2.8}$$

which will be used in our proofs of main theorems later.

Since K satisfies (**PCC**), we could find $0 < r_\theta < 1/2$ fixed such that $B_\theta := B_{r_\theta |x_0|}(x_0)$ lying in the cone where$|K(x)| \ge c_1 |x|^\gamma$ as $|x_0| \to \infty$ suitably. With help of Fubini's theorem, direct computation yields that

$$\begin{aligned} &\left| \int_{B_\theta} \int_{|x-y| \le 1} \log \frac{1}{|x-y|} K e^{nw} \mathrm{d}y \mathrm{d}x \right| \\ &\quad \le \int_{B_\theta} \int_{|x-y| \le 1} \frac{1}{|x-y|} |K| e^{nw} \mathrm{d}y \mathrm{d}x \\ &\quad \le \int_{B_{r_\theta |x_0| + 1}(x_0)} |K| e^{nw} \int_{B_\theta} \frac{1}{|x-y|} \mathrm{d}x \mathrm{d}y \\ &\quad \le C |x_0|^{n-1}. \end{aligned}$$

Meanwhile, notice that $1 - r_\theta \le \frac{|x|}{|x_0|} \le 1 + r_\theta$ for $x \in B_\theta$. Thus

$$\frac{1}{|B_\theta|} \int_{B_\theta} w(x) \mathrm{d}x = (-\alpha + o(1)) \log |x_0|$$

where we use the fact $w(x) = \int_{\mathbb{R}^n} \log \frac{|y|}{|x-y|} K e^{nw} \mathrm{d}y$. With help of Jensen's inequality, we have

$$e^{\frac{1}{|B_\theta|} \int_{B_\theta} n w(x) \mathrm{d}x} \le \frac{1}{|B_\theta|} \int_{B_\theta} e^{nw} \mathrm{d}x.$$

Recall $|K| \geq c_1|x|^\gamma$ in B_θ and then one has

$$|x_0|^{n+\gamma-n\alpha+o(1)} \leq C \int_{B_\theta} |K|e^{nw}\mathrm{d}x.$$

Notice that the right side of above inequality tends to zero as $|x_0| \to \infty$ based on $Ke^{nw} \in L^1(\mathbb{R}^n)$. Thus we have

$$\alpha \geq 1 + \frac{\gamma}{n} \geq 2. \tag{2.9}$$

Since $x \cdot \nabla K \leq 0$, there holds

$$K(x) - K(0) = \int_0^1 x \cdot \nabla K(tx)\mathrm{d}t = \int_0^1 \frac{tx \cdot \nabla K(tx)}{t}\mathrm{d}t \leq 0$$

which yields that $K(x) \leq C$. For brevity, we utilize φ^+ and φ^- to represent the positive and negative parts of the function φ, respectively. Combing with $w \leq C$, we get the bound

$$0 \leq \int_{A_1} \log\frac{1}{|x-y|}K^+e^{nw}\mathrm{d}y \leq C. \tag{2.10}$$

Then we rewrite (2.7) as

$$w(x) = (-\alpha + o(1))\log|x| - \frac{2}{\omega_n}\int_{A_1}\log\frac{1}{|x-y|}K^-e^{nw}\mathrm{d}y. \tag{2.11}$$

Immediately, $w(x) \leq (-\alpha + o(1))\log|x|$. Since $\alpha \geq 1 + \frac{\gamma}{n}$ and $|K| \leq C|x|^\gamma$ for sufficiently large $|x|$, we have

$$|K|e^{nw} \leq C$$

and then

$$\left|\frac{2}{\omega_n}\int_{A_1}\log\frac{1}{|x-y|}K^-e^{nw}\mathrm{d}y\right| \leq C.$$

for sufficiently large $|x|$. Finally, we have

$$\lim_{|x|\to\infty}\frac{w(x)}{\log|x|} = -\alpha. \tag{2.12}$$

Now we are going to give a more precise upper bound of w as following

$$w(x) \leq -\alpha\log|x| + C.$$

Firstly, direct computation yields that

$$\begin{aligned} & w(x)+\alpha \log |x| \\ & \quad=\frac{2}{\omega_n} \int_{\mathbb{R}^n} \log \frac{|x| \cdot(|y|+1)}{|x-y|} K^{+} e^{n w} \mathrm{d} y \\ & \quad+\frac{2}{\omega_n} \int_{\mathbb{R}^n} \log \frac{|x| \cdot(|y|+1)}{|x-y|}\left(-K^{-}\right) e^{n w} \mathrm{d} y \\ & \quad+\frac{2}{\omega_n} \int_{\mathbb{R}^n} \log \frac{|y|}{|y|+1} K e^{n w} \mathrm{d} y \\ & \quad=: I_1+I_2+I_3 \end{aligned}$$

For $|x| \geq 1$, it is easy to check

$$\frac{|x| \cdot(|y|+1)}{|x-y|} \geq 1$$

and then

$$I_2 \leq 0.$$

Based on our assumptions, one has $I_3=C_0$ for some constant.

As for the term I_1, we need use again the fact $K^{+} \leq C$. By using (2.12) and (2.9), we have

$$e^{n w(y)} \leq C|y|^{-\frac{3 n}{2}} \tag{2.13}$$

for $|y| \geq 1$. We split I_1 as

$$\begin{aligned} I_1= & \int_{|x-y| \leq \frac{|x|}{2}} \log \frac{|x| \cdot(|y|+1)}{|x-y|} K^{+} e^{n w} \mathrm{d} y \\ & +\int_{|x-y| \geq \frac{|x|}{2}} \log \frac{|x| \cdot(|y|+1)}{|x-y|} K^{+} e^{n w} \mathrm{d} y \\ = & : I I_1+I I_2 . \end{aligned}$$

We deal with these terms one by one. Notice that $|x|/2 \leq |y| \leq 3|x|/2$ when $|x-y| \leq |x|/2$. Then

$$\begin{aligned} I I_1= & \int_{|x-y| \leq \frac{|x|}{2}} \log (|x|(|y|+1)) K^{+} e^{n w} \mathrm{d} y+\int_{|x-y| \leq \frac{|x|}{2}} \log \frac{1}{|x-y|} K^{+} e^{n w} \mathrm{d} y \\ \leq & C \log \left(|x|\left(\frac{3|x|}{2}+1\right)\right)|x|^{-3 n / 2} \int_{|x-y| \leq|x| / 2} \mathrm{d} y \\ & +C|x|^{-3 n / 2} \int_{|x-y| \leq|x| / 2} \frac{1}{|x-y|} \mathrm{d} y \\ \leq & C \log \left(3|x|^2 / 2+|x|\right)|x|^{-n / 2}+C|x|^{-n / 2+1} \leq C \end{aligned}$$

for $|x| >> 1$. As for the term II_2, we make use of $|x|/|x-y| \leq 2$ and (2.13) to get

$$II_2 \leq \int_{|x-y|\geq \frac{|x|}{2}} \log(2|y|+2)K^+e^{nw}\mathrm{d}y \leq C.$$

Combining above estimates, we finally get the desired results

$$w(x) + \alpha \log|x| \leq C$$

for $|x| >> 1$. □

Combining Lemma 2.1 and Lemma 2.2, it seems that we have got our desired result based on the assumption $x \cdot \nabla K \leq 0$. Actually, if we just consider the solution of the integral equation, we have gotten the non-existence theorem which generalizes a result of Hyder and Martinazzi [14].

Theorem 2.3 (Proposition 2.8 in [14]) *For* $K(x) = 1 - |x|^p,\ \ x \in \mathbb{R}^4,\ p \geq 4$, *there is no solution to the integral equation*

$$u(x) = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} \log\left(\frac{|y|}{|x-y|}\right) K(y)e^{4u(y)}\mathrm{d}y + c$$

for some $c \in \mathbb{R}$ *with* $Ke^{4u} \in L^1(\mathbb{R}^4)$.

Actually, in this sepcial case $K(x) = 1 - |x|^p$ with $p \geq 4$, it is easy to check that $x \cdot \nabla K \leq 0$ and $K(x)$ satisfies (**PCC**). As for the harmless constant c, we can just do a scaling over K like the strategy taken in the proof of Theorem 1.4 such that $c = 0$. It is obvious that the conditions $x \cdot \nabla K \leq 0$ and (**PCC**) still hold after a scaling. With help of Lemmas 2.1 and 2.2, we have the following corollary.

Corollary 2.4 *Given a function* $K(x) \in C^1(\mathbb{R}^n)$ *satisfying* (**PCC**) *and* $x \cdot \nabla K(x) \leq 0$. *There is no solution to the integral Eq. (2.1) with* $Ke^{nu} \in L^1(\mathbb{R}^n)$, $u \in L^\infty_{loc}(\mathbb{R}^n)$ *and* $u \leq C$.

Remark 2.5 We should point out that we assume $u \leq C$ in addition compared with Theorem 2.3. Actually, if K^+ has compact support, we can remove this assumption by noticing that the estimate (2.10) is trivial in this case. With help of $u \leq C$, we could deal with the degenerate cases such as $K(x) = 1 - x_1^n$.

However, as for the conformal equation itself, we still need to study the connection between the integral equation and the conformal equation. Before we finish our proof of main theorems, we establish the following two useful lemmas for readers' convenience.

Lemma 2.6 *Suppose* $p(x)$ *is a polynomial on* $\mathbb{R}^n$ *and* $deg(p(x)) \geq n$, *then* $p(x)$ *satisfies* (**PCC**).

Proof Denote $k := deg(p(x)) \geq n$ and separate $p(x)$ as

$$p(x) = p_k(x) + l(x)$$

where $p_k(x)$ is a homogeneous polynomial of degree k and $\deg(l(x)) \leq k-1$. We choose the polar coordinate (r, θ) with $r \geq 0$ and $\theta \in \mathbb{S}^{n-1}$. Then one has

$$p_k(x) = r^k \varphi_k(\theta)$$

where $\varphi_k(\theta)$ is a non-zero smooth function defined on $\mathbb{S}^{n-1}$. There exist $c_1 > 0$ and a geodesic ball $B_{r_1}(\theta_0) \subset \mathbb{S}^{n-1}$ such that

$$|\varphi_k(\theta)| \geq 2c_1 > 0$$

for any $\theta \in B_{r_1}(\theta_0)$. Then

$$|p(x)| \geq 2c_1|x|^k - c_2|x|^{k-1}$$

where $c_2 > 0$ is a constant depending only on the cofficients of $l(x)$. We choose $R_1 = \max\{1, \frac{c_2}{c_1}\}$ and then one has

$$|p(x)| \geq c_1|x|^k + |x|^{k-1}(c_1|x| - c_2) \geq c_1|x|^k \geq c_1|x|^n$$

for any $x \in [R_1, +\infty) \times B_{r_1}(\theta_0)$. □

For readers' convenience, we slightly improve Theorem 5 in [20] to be used later which is a generalization of Theorem 2.3 in [1] with help of Pizzetti's formula [24].

Lemma 2.7 (Theorem 5 in [20]) *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \leq C(1+|x|^l)$, for some $l \geq 0$. Then $h(x)$ is a polynomial of degree at most* $\max\{l, 2m-2\}$.

Proof If $l \geq 2m-2$, there is nothing to do because this is exactly the result in [20] which concludes that $h(x)$ is a polynomial of degree at most l. If $l < 2m-2$, then we simply bound h from above by a multiple of $1 + |x|^{2m-2}$ which returns to the former case. Thus $h(x)$ is a polynomial of degree at most $\max\{l, 2m-2\}$. □

3 Proof of Theorems

Proof of Theorem 1.4 Following Chen-Li [6], we introduce

$$\tilde{u}(x) = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|y|}{|x-y|} K(y)e^{2u}\mathrm{d}y.$$

Then the function $v := u - \tilde{u}$ is harmonic. By using mean value property of harmonic function, one has

$$v(x_0) = \frac{1}{|B_1(x_0)|}\int_{B_1(x_0)} (u(x) - \tilde{u}(x))\mathrm{d}x \leq C + C\log(2 + |x_0|)$$

where we have used the estimate (2.8) and the assumption $u \leq C$. Therefore, v must be a constant. This is a very classical result, which can be also obtained from Lemma 2.7 by choosing $m = 1$ and $l = 1$. It is not hard to check $v(x) \leq C(1 + |x|)$. Then $v(x)$ is a polynomial of degree at most one. However, the estimate $v(x) \leq C + C\log(2+|x|)$ rules out the case degree of one. Hence $\tilde{u} = u + C_1$ for $C_1 \in \mathbb{R}$. Then

$$\tilde{u} = \frac{1}{2\pi}\int_{\mathbb{R}^2} \log\frac{|y|}{|x-y|} K(y)e^{-2C_1}e^{2\tilde{u}}\mathrm{d}y$$

satisfying the integral Eq. (2.1). Then $K(y)e^{-2C_1}e^{2\tilde{u}} \in L^1(\mathbb{R}^2)$ and $K(y)e^{-2C_1}$ satisfies (**PCC**) which shows that

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\langle x, \nabla K(x)\rangle e^{2u(x)}\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2}\langle x, \nabla(K(x)e^{-2C_1})\rangle e^{2\tilde{u}(x)}\mathrm{d}x = \alpha(\alpha-2)$$

where $\alpha = \frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{-2C_1}e^{2\tilde{u}}\mathrm{d}x = \frac{1}{2\pi}\int_{\mathbb{R}^2} K(x)e^{2u}\mathrm{d}x$. Since $x \cdot \nabla K \leq 0$ and $K(x)$ satisfying (**PCC**), we have

$$\frac{1}{2\pi}\int_{\mathbb{R}^2}\langle x, \nabla K(x)\rangle e^{2u(x)}\mathrm{d}x < 0$$

which concludes that $0 < \alpha < 2$ contradicting to Lemma 2.2. □

Now we imitate the proof of Theorem 1.4 to give the proof of higher order case.

Proof of Theorem 1.5 Following the argument in [27], we consider

$$\tilde{u} = \frac{\alpha_n}{\omega_n}\int_{\mathbb{R}^n}\left(\log\frac{|y|}{|x-y|}\right) K(y)e^{nu(y)}\mathrm{d}y$$

where α_n is a positive constant such that

$$(-\Delta)^{n/2}\tilde{u} = K(x)e^{nu(x)}.$$

Actually, $\tilde{u}$ is also smooth based on the smoothness assumption of $K(x)$ and $u(x)$ as well as $Ke^{nu} \in L^1(\mathbb{R}^n)$. Following the argument of the proof of Theorem 1.2 in [16], we consider $B_4(x_0)$ for any $x_0 \in \mathbb{R}^n$. Choose a smooth cut-off function $0 \leq \eta(x) \leq 1$ such that $\eta(x) = 1$ in $B_2(x_0)$ and vanishes outside $B_4(x_0)$. For $x \in B_1(x_0)$, we can split $\tilde{u}$ by

$$\begin{aligned}\tilde{u}(x) &= \frac{\alpha_n}{\omega_n}\int_{\mathbb{R}^n}\left(\log\frac{|y|}{|x-y|}\right)\eta(y)K(y)e^{nu(y)}\mathrm{d}y + \frac{\alpha_n}{\omega_n}\int_{\mathbb{R}^n}\left(\log\frac{|y|}{|x-y|}\right)\\ &\quad (1-\eta(y))K(y)e^{nu(y)}\mathrm{d}y\\ &= \frac{\alpha_n}{\omega_n}\int_{\mathbb{R}^n}\left(\log\frac{|y|}{|x-y|}\right)\eta(y)K(y)e^{nu(y)}\mathrm{d}y + \frac{\alpha_n}{\omega_n}\int_{\mathbb{R}^n\setminus B_2(x_0)}\left(\log\frac{|y|}{|x-y|}\right)\\ &\quad (1-\eta(y))K(y)e^{nu(y)}\mathrm{d}y\end{aligned}$$

$$=: I(x) + II(x)$$

Since $\eta(x)K(x)e^{nu} \in C_c^\infty(\mathbb{R}^n)$, with help of Theorem 10.3 in [18], we have $I(x) \in C^\infty(B_1(x_0))$. Since $(1-\eta)Ke^{nu} \in L^1(\mathbb{R}^n)$, we can differentiate $II(x)$ under the integral sign for $x \in B_1(x_0)$, so $II(x) \in C^\infty(B_1(x_0))$. Thus $\tilde{u} \in C^\infty(\mathbb{R}^n)$. Set $v = u - \tilde{u}$ and then $(-\Delta)^{n/2}v = 0$.

If we show that v is a constant, combining Lemmas 2.1 and 2.2 will show the contradiction as the proof of Theorem 1.4.

Direct computation yields that

$$\begin{aligned}
\left|\int_{B_1(x_0)} (-\Delta)\tilde{u}\mathrm{d}x\right| =& \frac{\alpha_n}{\omega_n}\left|\int_{B_1(x_0)}\int_{\mathbb{R}^n} (-\Delta)_x\left(\log\frac{1}{|x-y|}\right)K(y)e^{u(y)}\mathrm{d}y\mathrm{d}x\right| \\
\le& C\int_{\mathbb{R}^n}\int_{B_1(x_0)} \frac{|K(y)|e^{nu(y)}}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\
=& C\int_{B_{|x_0|/2}(0)}\int_{B_1(x_0)} |K(y)|\frac{e^{nu(y)}}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\
&+ C\int_{\mathbb{R}^n\backslash(B_{|x_0|/2}(0)\cup B_2(x_0))}\int_{B_1(x_0)} |K(y)|\frac{e^{nu(y)}}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\
&+ C\int_{B_2(x_0)}\int_{B_1(x_0)} |K(y)|\frac{e^{nu(y)}}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\
\le& C\left(\frac{|x_0|}{2}-1\right)^{-2} + C\int_{\mathbb{R}^n\backslash B_{|x_0|/2}(0)} K(y)e^{nu(y)}\mathrm{d}y \\
&+ C\int_{B_2(x_0)}\int_{B_4(y)} |K(y)|\frac{e^{nu(y)}}{|x-y|^2}\mathrm{d}x\mathrm{d}y \\
\le& C\left(\frac{|x_0|}{2}-1\right)^{-2} + C\int_{\mathbb{R}^n\backslash B_{|x_0|/2}(0)} K(y)e^{nu(y)}\mathrm{d}y \\
&+ C\int_{B_2(x_0)} |K(y)|e^{nu(y)}\mathrm{d}y \to 0 \quad \text{as } |x_0| \to \infty.
\end{aligned}$$

Combing with the assumption (1.4), we have

$$\left|\int_{B_1(x_0)} \Delta v\mathrm{d}x\right| \to 0, \quad \text{as } |x_0| \to \infty. \tag{3.1}$$

With help of (2.8), one has

$$\int_{B_1(x_0)} v^+\mathrm{d}x \le \int_{B_1(x_0)} \left(u^+ + |\tilde{u}|\right)\mathrm{d}x \le C + C\log(2+|x_0|). \tag{3.2}$$

If x_0 fixed and $R >> |x_0|$, we could choose suitable $B_1(x_i)$ where $x_i \in B_{2R}(0)$ such that $B_R(x_0) \subset \cup_{i=1}^N B_1(x_i)$ and $B_{1/5}(x_i)$ are disjoint as well as $\cup_i^N B_{1/5}(x_i) \subset B_{3R}(0)$.

Then

$$\begin{aligned}\frac{1}{|B_R(x_0)|}\int_{B_R(x_0)} v^+(y)\mathrm{d}y \leq& \frac{1}{|B_R(x_0)|}\sum_{i=1}^{N}\int_{B_1(x_i)} v^+\mathrm{d}x\\ \leq& C\frac{1}{|B_R(x_0)|}\sum_{i=1}^{N}\int_{B_1(x_i)}\log(2R+2)\mathrm{d}x\\ =& C\frac{5^n}{|B_R(x_0)|}\int_{\cup_i^N B_{1/5}(x_i)}\mathrm{d}x\log(2R+2)\\ \leq& C\log(2R+2)\end{aligned}$$

Following the argument of Theorem 5 in [20] and using Propostion 4 in [20], we have for any $x\in\mathbb{R}^n$ fixed and $R >> |x|$,

$$\begin{aligned}|D^{n-1}v(x)| \leq& \frac{C}{R^{n-1}}\frac{1}{|B_R(x)|}\int_{B_R(x)}|v(y)|\mathrm{d}y\\ =& \frac{C}{R^{n-1}}\left(-\frac{1}{|B_R(x)|}\int_{B_R(x)} v(y)\mathrm{d}y + 2\frac{1}{|B_R(x)|}\int_{B_R(x)} v^+(y)\mathrm{d}y\right)\\ \leq& \frac{C}{R^{n-1}}\left(\sum_{i=0}^{n/2-1} c_i R^{2i}\Delta^i h(x) + \log(2R+2)\right)\\ =& O(R^{-1})\end{aligned}$$

where we use Pizzetti's formula for polyhamonic function $\Delta^{n/2}v = 0$ (See Lemma 3 in [20]). Then by taking $R\to\infty$, one has $D^{n-1}v(x) = 0$ which implies that v is a polynomial of degree at most $n-2$.

Then Δv is a polynomial of degree at most $n-4$. Applying the estimate (3.1) to get that $\Delta v = 0$. Then recall the estimate (3.2) to get that v must be a constant similar to the argument in the proof of Theorem 1.4. In fact, using mean value property and the estimate (3.2), one has

$$v(x_0) = \frac{1}{|B_1(x_0)|}\int_{B_1(x_0)} v(x)\mathrm{d}x \leq \frac{1}{|B_1(x_0)|}\int_{B_1(x_0)} v^+(x)\mathrm{d}x \leq C + C\log(|x_0|+2).$$

Then Lemma 2.7 yields that v must be a constant. The remaining argument is the same as that of Theorem 1.4. Finally, we complete our proof. □

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Data availability The authors confirms that data supporting the findings of this study are available within the article.

A Appendix

Lemma A.1 *Suppose n is an even integer and $K(x) \in C^{k,\alpha}_{loc}(\mathbb{R}^n)$, with $k \geq 0$ and $0 < \alpha < 1$, each solution to the intergal equation*

$$u(x) = \frac{2}{\omega_n} \int_{\mathbb{R}^n} \left(\log \frac{|y|}{|x-y|} \right) K(y) e^{nu(y)} \mathrm{d}y \tag{A.1}$$

with $Ke^{nu} \in L^1(\mathbb{R}^n)$ and $u \in L^\infty_{loc}(\mathbb{R}^n)$, then u belongs to $C^{n+k,\alpha}_{loc}(\mathbb{R}^n)$. In particularly, if $K(x)$ is smooth, so is $u(x)$.

Proof Given any $B_4(x_0) \subset \mathbb{R}^n$, choose $R = 2|x_0| + 8$ and split the right side of the integral equation as

$$\begin{aligned}
&\frac{2}{\omega_n} \int_{\mathbb{R}^n} \left(\log \frac{|y|}{|x-y|} \right) K(y) e^{nu(y)} \mathrm{d}y \\
&\quad = \frac{2}{\omega_n} \int_{B_R(0)} \left(\log \frac{|y|}{|x-y|} \right) K(y) e^{nu(y)} \mathrm{d}y \\
&\qquad + \frac{2}{\omega_n} \int_{\mathbb{R}^n \backslash B_R(0)} \left(\log \frac{|y|}{|x-y|} \right) K(y) e^{nu(y)} \mathrm{d}y \\
&\quad =: I(x) + II(x)
\end{aligned}$$

inspired by the proof of Theorem 1.2 in [16]. It is not hard to check $|\log \frac{|y|}{|x-y|}| \leq C$ for $x \in B_4(x_0)$ and $y \in \mathbb{R}^n \backslash B_R(0)$. Thus, $II(x)$ is well-defined for $x \in B_4(x_0)$ with help of $Ke^{nu} \in L^1(\mathbb{R}^n)$. Since we can can differentiate $II(x)$ under the integral sign for $x \in B_4(x_0)$, so $II(x) \in C^\infty(B_4(x_0))$. For some constant γ_n, $\frac{1}{\gamma_n} \log \frac{1}{|x|}$ is a fundamental solution of the operator $(-\Delta)^{n/2}$ in $\mathbb{R}^n$ satisfying

$$(-\Delta)^{n/2} \left(\frac{1}{\gamma_n} \log \frac{1}{|x|} \right) = \delta_0, \quad \text{in } \mathbb{R}^n. \tag{A.2}$$

Due to (A.2), it is easy to see

$$(-\Delta)^{n/2} II(x) = 0, x \in B_4(x_0). \tag{A.3}$$

Since $K \in C^{k,\alpha}_{loc}(\mathbb{R}^n)$ and $u \in L^\infty_{loc}(\mathbb{R}^n)$, both $\log|y| K(y) e^{nu(y)}$ and $\log|x-y| K(y) e^{nu(y)}$ are integrable over $B_R(0)$ for any $x \in B_4(x_0)$. Then

$$I(x) = \int_{\mathbb{R}^n} \log \frac{1}{|x-y|} K(y) e^{nu(y)} \chi_{B_R(0)}(y) \mathrm{d}y + c_1$$

where c_1 is a constant. With help of Theorem 6.21 in [18] and the fundamental solution (A.2), we have

$$(-\Delta)^{n/2} I(x) = \gamma_n K e^{nu}, \ x \in B_4(x_0)$$

in the sense of distribution. Combing with (A.3), one has

$$(-\Delta)^{n/2}u(x) = \gamma_n K e^{nu}, \ x \in B_4(x_0)$$

in sense of distribution.

Now, following the argument of Corollary 8 in [20], we write $u|_{B_4(x_0)} = v_1 + v_2$, where

$$\begin{cases} (-\Delta)^{n/2}v_1 = \gamma_n K e^{nu} & \text{in } B_4(x_0) \\ v_1 = \Delta v_1 = \cdots = \Delta^{n/2-1}v_1 = 0 & \text{on } \partial B_4(x_0) \end{cases}$$

and $\Delta^{n/2}v_2 = 0$. It is easy to see $Ke^{nu} \in L^\infty(B_4(x_0))$ due to $K \in C^{k,\alpha}_{loc}$ and $u \in L^\infty_{loc}$. Hence, for any $p > 0$, one has $v_1 \in W^{n,p}(B_4(x_0))$. Then for sufficienty large p, Sobolev embedding theorem yields that $v_1 \in C^{n-1,\beta}(B_4(x_0))$ for some $0 < \beta < 1$, while v_2 is smooth in $B_4(x_0)$. Thus $u \in C^{n-1,\beta}(B_2(x_0))$. Then with the same procedure, we can bootsrap and make use of Schauder's estimate to prove that $u \in C^{n+k,\alpha}(B_1(x_0))$ see Theorem 9.19 in [13]. □

Lemma A.2 *When $p(x)$ is a polynomial with $deg(p) = k \geq 2$ and $p(x) = p_k(x) + l(x)$ with $p_k(x)$ is a homogeneous polynomial of degree k and $deg(l) \leq k-1$. Suppose $|p_k(x)| \neq 0$ for any $x \neq 0$ and $u \leq C$, the following two conditions are equivalent:*

(a) $\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x < +\infty$ *and* $\int_{\mathbb{R}^n} p(x)e^{nu}\mathrm{d}x \in \mathbb{R}$
(b) $\int_{\mathbb{R}^n} |p(x)|e^{nu}\mathrm{d}x < +\infty.$

Proof Firstly, we show that condition (a) implies (b). Without loss of generality, suppose $p_k(x) > 0$ for any $x \neq 0$. There exists a constant C_1 such that

$$|l(x)| \leq \frac{1}{2}p_k + C_1$$

with help of the definitions of l and p_k as well as Hölder's inequality. Then

$$\begin{aligned} \int_{\mathbb{R}^n} p_k(x)e^{nu}\mathrm{d}x &= \int_{\mathbb{R}^n} p(x)e^{nu}\mathrm{d}x - \int_{\mathbb{R}^n} l(x)e^{nu}\mathrm{d}x \\ &\leq \int_{\mathbb{R}^n} p(x)e^{nu}\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^n} p_k(x)e^{nu}\mathrm{d}x + C_1\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x \end{aligned}$$

Thus

$$\int_{\mathbb{R}^n} p_k(x)e^{nu}\mathrm{d}x \leq 2\int_{\mathbb{R}^n} p(x)e^{nu}\mathrm{d}x + 2C_1\int_{\mathbb{R}^n} e^{nu}\mathrm{d}x < +\infty.$$

Now, we have

$$\int_{\mathbb{R}^n} |p(x)|e^{nu}\mathrm{d}x \leq \int_{\mathbb{R}^n} p_k e^{nu}\mathrm{d}x + \int_{\mathbb{R}^n} |l(x)|e^{nu}\mathrm{d}x$$

$$\leq \frac{3}{2} \int_{\mathbb{R}^n} p_k(x) e^{nu} \mathrm{d}x + C_1 \int_{\mathbb{R}^n} e^{nu} \mathrm{d}x < +\infty.$$

Conversely, if $\int_{\mathbb{R}^n} |p(x)| e^{nu} \mathrm{d}x < +\infty$, it is obvious that $\int_{\mathbb{R}^n} p(x) e^{nu} \mathrm{d}x \in \mathbb{R}$. For some sufficiently large R, there exists a constant $C_2 > 0$ such that $|p(x)| \geq C_2$ over $B_R(0)^c$. Then we have

$$\int_{B_R(0)^c} e^{nu} \mathrm{d}x \leq \frac{1}{C_2} \int_{B_R(0)^c} |p(x)| e^{nu} \mathrm{d}x < +\infty.$$

Due to the assumption $u \leq C$, one has

$$\int_{B_R(0)} e^{nu} \mathrm{d}x \leq e^{nC} |B_R(0)| < +\infty.$$

Combing these two estimates, we conclude that

$$\int_{\mathbb{R}^n} e^{nu} \mathrm{d}x < +\infty.$$

□

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