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Asymptotic Behavior of Conformal Metrics with Null Q-curvature

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Abstract
We describe the asymptotic behavior of conformal metrics related to the GJMS operator in the null case, as the prescribed Q-curvature $f_0(x) + \lambda$ gradually changes. We show that if one of the maximum points of f_0 is flat up to order $n - 1$, the normalized conformal metrics in the lowest energy level will form exactly one spherical bubble as λ approaches zero using higher order Bol's inequality. This generalizes the result of Struwe (J Eur Math Soc 22:3223-3262, 2020) in the two-dimensional case to higher dimensions and helps rule out the slow bubble case discussed by Ngô and Zhang (Bubbling of the prescribed Q-curvature equation on 4-manifolds in the null case. arXiv:1903.12054) to some degree.

Keywords Q-curvature · Asymptotic behavior · Higher order Bol's inequality

Mathematics Subject Classification 58E30 · 35J60 · 35J35

1 Introduction

Given a compact Riemann surface (M^2, g), the prescribed Gaussian curvature problem in conformal geometry attracts a lot of interest. It is equivalent to solve the following conformal equation

$$-\Delta_g u + K_g = f e^{2u} \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator, K_g is Gaussian curvature of (M^2, g) and f is the prescribed smooth function defined on M^2. The special case where the function f is a constant has been extensively studied due to the uniformization theorem. In the more general case, the problem becomes more complex and has been the subject of ongoing research in the field.

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The seminal work of Kazdan and Warner [17] has played a crucial role in the development of the prescribed curvature problem and we highly recommend referring to it for further details. The Eq. (1.1) is well known as the Nirenberg problem when (M^2, g) is a standard sphere. In their seminal work [17], Kazdan and Warner provided an obstruction, known as Kazdan-Warner's identity, to the existence of solutions to (1.1), which significantly increases the difficulty of the Nirenberg problem. Many important contributions have been made towards the solution of this problem, and we refer the interested reader to [6, 7], and references therein.

In the case of a compact four-dimensional manifold (M^4, g), Paneitz [28] introduced a conformal operator defined by

$$P_g^4 = \Delta_g^2 - \mathrm{div}_g \left(\left(\frac{2}{3} R_g g - 2\mathrm{Ric}_g \right) d \right)$$

where R_g and Ric_g denote the scalar curvature and Ricci curvature tensor of (M^4, g), respectively. Later, Branson [4] introduced the Q-curvature defined by

$$Q_g = -\frac{1}{12}(\Delta_g R_g - R_g^2 + 3|\mathrm{Ric}_g|^2).$$

Interestingly, similar to the Eq. (1.1), the prescribed Q-curvature problem in the conformal class of (M^4, g) is equivalent to solving the following equation

$$P_g^4 u + Q_g^4 = f e^{4u} \tag{1.2}$$

for the conformal metric $\tilde{g} = e^{2u} g$ where $Q_g^4 = 2Q_g$. For higher order cases, Graham, Jenne, Mason and Sparling [15] introduced an operator known as GJMS operator P_g^n for (M^n, g) satisfying

$$P_g^n u + Q_g^n = Q_{\tilde{g}}^n e^{nu} \tag{1.3}$$

where $\tilde{g} = e^{2u} g$ where $n \geq 4$ is an even integer. We refer the interested reader to [8, 11, 23, 25], and the references therein for more details.

In the current paper, our focus is on the null case, i.e., $Q_g^n \equiv 0$, and we aim to describe the asymptotic behavior of a family of conformal metrics. To facilitate better understanding, we begin with the two-dimensional case. When the background Gaussian curvature vanishes, Kazdan and Warner [17] provided a sufficient and necessary condition for the existence of solutions to (1.1):

$$\sup_M f > 0 \text{ and } \int_M f \mathrm{d}\mu_g < 0 \quad \text{or} \quad f \equiv 0. \tag{1.4}$$

It is natural to ask what will happen to the conformal metrics if this condition is gradually violated? Let us explain it more precisely. Consider a non-constant smooth function $f_0 \leq 0$ on M that vanishes at some points. Given a constant $\lambda \in (0, -\bar{f}_0)$ where $\bar{f}_0 := \frac{1}{vol(M)} \int_M f_0 \mathrm{d}\mu_g$, it is easy to check that this new function $f_\lambda := f_0 +$

λ satisfies Kazdan-Warner's condition (1.4). With help of the method in [17], the solutions to (1.1) exist by choosing the minimizer denoted by u_λ of the Dirichlet energy $\int_M |\nabla_g u|^2 \mathrm{d}\mu_g$ under the constraint that $u \in H^1(M, g)$ and $\int_M f_\lambda e^{2u} \mathrm{d}\mu_g = 0$. For brevity, we denote the Dirichlet energy of the minimizer u_λ as follows

$$\beta(\lambda) := \int_M |\nabla_g u_\lambda|^2 \mathrm{d}\mu_g.$$

When λ tends to 0 or $-\bar{f}_0$, we find that f_λ gradually violates the Kazdan-Warner's condition (1.4). Galimberti's study in [12] focused on the asymptotic behavior of u_λ where he found that as λ approaches zero, $\beta(\lambda)$ tends to infinity monotonically and the metrics blow up at finite points. Additionally, he provided several possible characterizations of the bubbles (see Theorem 1.1 in [12]). Moreover, Galimberti presented a complete description of the case where $\lambda \to -\frac{1}{\mathrm{vol}(M)} \int_M f_0 \mathrm{d}\mu_g$ and he demonstrated that $\beta(\lambda)$ tends to zero while the metrics collapse. Subsequently, Struwe [29] provided a more detailed characterization of the former case by ruling out the possibility of the "slow bubble" scenario:

$$-\Delta u = (1 + A(x, x))e^{2u}, x \in \mathbb{R}^2 \tag{1.5}$$

where A is a negative 2×2 matrix with $(1 + |A(x, x)|)e^{2u} \in L^1(\mathbb{R}^2)$ which is a possible case in Galimberti's theorem (see Theorem 1.1 in [12]). In fact, Struwe established a Liouville-type result (refer to Theorem 1.3 in [29]) that proved the nonexistence of a solution to (1.5). With help of this result, he then demonstrated that when f_0 is non-degenerate at each maximum point, the metrics can only form at most two spherical bubbles. This result was further improved by the author in [21], in collaboration with Xu, where we showed that the metrics will shape exactly one spherical bubble even in more general cases based on a global version of Alexandrov-Bol's inequality. Meanwhile, for the general Liouville-type theorem related to the Eq. (1.5) was established in [18]. For a similar phenomenon in compact surfaces of higher genus, there have been several studies in the literature, such as [3, 10], and [9], among others.

In the case of four-dimensional manifolds, Ngô and Zhang [26] studied a similar phenomenon related to Q-curvature based on an existence result for (1.2) (see Theorem 1.1 in [14]) in the null case. They gave a possible characterization in the fourth dimension similar to that in [12] and [13] when the maximum points of f_0 are non-degenerate. Different from the two dimensional case, Hyder and Martinazzi [16] showed that the following equation

$$\Delta^2 u(x) = (1 - |x|^l)e^{4u(x)}, \quad x \in \mathbb{R}^4 \tag{1.6}$$

with $(1 + |x|^l)e^{4u} \in L^1(\mathbb{R}^4)$ does have solutions for $0 < l < 4$. When $l \geq 4$, a Liouville-type theorem is established by [30] and [16] which shows that there is no solutions to (1.6) if we additionally assume that $\Delta u(x) \to 0$ as $|x| \to \infty$. Therefore, we cannot exclude the possibility of a 'slow bubble' using a Liouville-type theorem similar to the one discussed by Struwe in [29], which addresses the two-dimensional case.

It seems like that the ’slow bubble’ does exist if the function has non-degenerated maximum points due to the work of Hyder and Martinazzi [16]. However, we tend to believe that there are no ’slow bubble’. In fact, we will show that the metrics will shape exactly one spherical bubble in the higher order cases if one of the maximum points of f_0 is flat up to $n-1$ order. This means "slow bubble" doesn’t exist in this case. It is worth noting that we allow some maximum points to be non-degenerate. Here, the flatness condition is only used to construct a test function to obtain the sharp upper bound of energy in Lemma 2.4. With help of such sharp bound and the higher order Bol’s inequality developed in [20], we are able to rule out the ’slow bubble’.

For convenience, we give some definitions before stating our main theorem.

Definition 1.1 We say a maximum point P_0 of $f \in C^\infty(M)$ is of $l-$type if in a small neighborhood of P_0, considering the exponential map $\exp_{P_0} : B_\delta(0) \subset \mathbb{R}^n \to M$, $f \circ \exp_{P_0}(z)$ has the expansion

$$f \circ \exp_{P_0}(z) = f(P_0) + p_l(z) + O(|z|^{l+1})$$

where $p_l(z) \leq 0$ is a homogeneous polynomial of degree l and $p_l(z) \neq 0$ if $z \neq 0$.

Definition 1.2 We say $f \in C^\infty(M)$ only has isolated maximum points if each maximum point is $l-$type for some integer l and there are only finitely many such maximum points.

Definition 1.3 We say a sequence $\{u_k\} \subset H^{\frac{n}{2}}(M, g)$ forms a spherical bubble at P_0 if there exists a subsequence as well as $r_k \to 0$ and $x_k = \exp_{P_0}(z_k) \to P_0$ such that

$$u_k \circ \exp_P(r_k z + z_k) + \log r_k \to \log\left(\frac{2}{1+|z|^2}\right),$$

up to a translation, a scaling or adding some constants, in $H^{n/2}_{loc}(\mathbb{R}^n)$.

For the sake of convenience, we use the notation $B_R(p)$ to refer to an Euclidean ball in $\mathbb{R}^n$ centered at $p \in \mathbb{R}^n$ with a radius of R. Also, $|B_R(p)|$ refers to the volume of $B_R(p)$ in terms of the Euclidean metric. For a function $\varphi(x)$, the positive part of $\varphi(x)$ is denoted as $\varphi(x)^+$ and the negative part of $\varphi(x)$ is denoted as $\varphi(x)^-$. Set $⨍_E \varphi(x)\mathrm{d}x = \frac{1}{|E|}\int_E \varphi(x)\mathrm{d}x$ for any measurable set E. We denote by C a constant which may be different from line to line. $[s]$ denotes the largest integer not greater than s.

Suppose that (M, g) is a compact manifold of even dimension $n \geq 4$ and GJMS operator (denoted as P_g for simplicity) is positive with kernel consisting of constants and Q-curvature vanishes. Consider a class of smooth functions on M defined as

$$\mathcal{F} := \{f \in C^\infty(M) | f \leq 0 \text{ and } f \not\equiv 0, \exists\, x_0 \in M, f(x_0) = 0\}$$

Given $f_0 \in \mathcal{F}$, let $f_\lambda(x) = f_0 + \lambda$ where the constant $\lambda \in (0, -\bar{f_0})$ where $\bar{f_0} = \frac{1}{vol(M)} \int_M f_0 \mathrm{d}\mu_g$. We define the energy functional

$$E(u) := \int_M u P_g u \mathrm{d}\mu_g,$$

under the constraint

$$\mathcal{M}(\lambda) := \left\{ u \in H^{\frac{n}{2}}(M, g) | \int_M f_\lambda e^{nu} \mathrm{d}\mu_g = 0 \right\},$$
$$\mathcal{M}^*(\lambda) := \left\{ u \in \mathcal{M}(\lambda) | \int_M e^{nu} \mathrm{d}\mu_g = 1 \right\}$$

where $H^{\frac{n}{2}}(M, g)$ denotes the Hilbert space. Based on Theorem 2.1 below and the property $E(u + c) = E(u)$ for any constant c, a minimizer u_λ of $E(u)$ in $\mathcal{M}^*(\lambda)$ is achieved satisfying

$$P_g u_\lambda = \alpha_\lambda f_\lambda e^{nu_\lambda} \tag{1.7}$$

where $\alpha_\lambda > 0$ and set the same notation as before

$$\beta(\lambda) := E(u_\lambda).$$

Now, we state our theorem in the current paper as follows.

Theorem 1.4 *Consider a compact manifold (M, g) of even dimension $n \geq 4$. Suppose that the GJMS operator of (M, g) is positive with kernel consisting of constants and the Q-curvature vanishes. Given a smooth function $f_0 \in \mathcal{F}$ which only has isolated maximum points. If one of the maximum points of f_0 is l-type with $l \geq n$, there exists a sequence $\lambda_k \downarrow 0$ such that $\{u_{\lambda_k}\}$ shapes only one spherical bubble at some maximum point p_0 of the prescribed function f_0. Moreover, $u_{\lambda_k}(x) \to -\infty$ for any $x \in M \backslash \{p_0\}$ and the following equality holds*

$$\lim_{k\to\infty} \lambda_k \alpha_{\lambda_k} = \frac{n}{2} \lim_{k\to\infty} \frac{\beta(\lambda_k)}{\log(1/\lambda_k)} = (n-1)! |\mathbb{S}^n|$$

where $|\mathbb{S}^n|$ denotes the volume of unit sphere in $\mathbb{R}^{n+1}$.

Remark 1.5 If the maximum points of f_0 are all non-degenerate, we still do not know how to eliminate the possibility of slow bubbles. We tend to believe that there are no slow bubbles.

In [21], the Alexandrov-Bol's inequality was used to rule out the "slow bubble" in the two-dimensional case. For brevity, Alexandrov-Bol's inequality on $\mathbb{R}^2$ shows that if the Gaussian curvature has an upper bound, the volume has a sharp lower bound. More details can be found in [2, 20], and the references therein. In the context of higher-order cases, a natural inquiry arises concerning the existence of analogous

Bol's inequalities. In [20], the author and J. Wei provide a partial response to this question. Leveraging the findings established in [20], we can effectively eliminate the presence of slow bubbles.

Now, we will briefly outline the structure of this paper. In Sect. 2, we provide another proof of the existence result for null Q-curvature related to GJMS operator. Additionally, we demonstrate the monotonicity of the minimizing energy and its growth control with respect to λ. In Sect. 3, we carry out a blow-up analysis for a sequence of normalized minimizers of energy. Finally, in Sect. 4, we present a proof of our main Theorem 1.4 with help of higher order Bol's inequality.

2 Existence of Minimizer and Monotonicity of Energy

Firstly, given a sign-changing smooth function $f(x)$ on (M, g), we give a proof of the existence of the minimizer of energy $E(u)$ under the constraint

$$\mathcal{M}_f := \left\{ u \in H^{\frac{n}{2}}(M) | \int_M f e^{nu} \mathrm{d}\mu_g = 0 \right\}.$$

Actually, it has been proved by Ge and Xu in [14] firstly. Besides, Ngô and Zhang gave another proof in their appendix of [27]. Here, our proof has its own interest since the continuity trick during the proof of monotonicity of energy will play a very important role. Meanwhile, compared with Corollary 2.4 in [1], we remove the restriction

$$\int_M |\Delta_g u|^2 \mathrm{d}\mu_g < \sigma,$$

for some positive constant σ depending on (M, g).

Theorem 2.1 *Given a smooth compact manifold (M^n, g) with unit volume where $n \geq 4$ is an even integer. Suppose that P_g is a positive GJMS operator with kernel consisting of constant functions. For any smooth and sign-changing function $f(x)$, the minimizers of $E(u)$ with $u \in \mathcal{M}$ exist. If we choose a minimizer denoted as u_0, up to a constant, u_0 satisfies*

$$P_g u_0 = sgn(-\bar{f}) f e^{nu_0} \tag{2.1}$$

where $\bar{f} = \int_M f \mathrm{d}\mu_g$ and sgn is the sign function.

Proof The case $\bar{f} = 0$ is trivial since all constant functions are the minimizers lying in $\mathcal{M}_f$ as well as the assumption on the kernel of P_g.

As for $\bar{f} \neq 0$, we just need to consider the case $\bar{f} < 0$ since we can do the same following program by considering $-f$. Suppose $f(x_0) = \max_M f(x) > 0$ and then there exists $\delta > 0$ such that $f(x) \geq \frac{1}{2} f(x_0)$ for any $x \in B_\delta(x_0)$. We could choose a non-negative cut-off function $\eta(x)$ such that $\eta(x)$ equals to c in $B_{\delta/2}(x_0)$ to be chosen later and vanishes outside $B_\delta(x_0)$. Then direct computation yields that

$$\begin{aligned}\int_M fe^{n\eta}\mathrm{d}\mu_g &= \int_M f(e^{n\eta}-1)\mathrm{d}\mu_g + \int_M f\mathrm{d}\mu_g \\ &= \int_{B_\delta(x_0)} f(e^{n\eta}-1)\mathrm{d}\mu_g + \int_M f\mathrm{d}\mu_g \\ &\geq \int_{B_{\delta/2}(x_0)} \frac{f(x_0)}{2}(e^{nc}-1)\mathrm{d}\mu_g + \int_M f\mathrm{d}\mu_g > 0\end{aligned}$$

if we choose sufficiently large c. Set a family of functions

$$\varphi(t) = \int_M fe^{tn\eta}\mathrm{d}\mu_g$$

where $t \in [0, 1]$ and then immediately $\varphi(1) > 0$ and $\varphi(0) < 0$. It is not hard to check that $\varphi(t)$ is continuous with respect to t. Then there exists $t_0 \in (0, 1)$ such that $\varphi(t_0) = 0$ which concludes that $t_0\eta \in \mathcal{M}_f$. Thus we show that $\mathcal{M}_f$ is not empty.

Due to positive P_g, for $u \in \mathcal{M}_f$, $E(u)$ is bounded from below by zero. Noticing that the fact that for any constant c, there holds

$$E(u + c) = E(u).$$

Thus we can normalize $\mathcal{M}_f$ as

$$\mathcal{M}_f^* = \left\{u \in \mathcal{M}_f | \int_M e^{nu}\mathrm{d}\mu_g = 1\right\}.$$

Thus we have

$$\mathcal{A} := \inf_{u\in\mathcal{M}_f} E(u) = \inf_{u\in\mathcal{M}_f^*} E(u). \tag{2.2}$$

Choose a sequence $\{u_i\} \subset \mathcal{M}_f^*$ satisfying $E(u_i) \leq C_0$ minimizing $E(u)$ where C_0 is a constant independent of i. Jensen's inequality implies that

$$\bar{u}_i := \int_M u_i\mathrm{d}\mu_g \leq \frac{1}{n}\log\int_M e^{nu_i}\mathrm{d}\mu_g = 0.$$

Applying Adams-Fontana inequality (see [8, 25]) and the fact that $E(u_i) \leq C_0$, there holds

$$-n\bar{u}_i = \log\int_M e^{n(u_i-\bar{u}_i)}\mathrm{d}\mu_g \leq C\int_M u_iP_gu_i\mathrm{d}\mu_g + C \leq C.$$

Combing with above two estimates, one has

$$|\bar{u}_i| \leq C.$$

Based on the positivity assumption on P_g as well as interpolation inequality, we have

$$\|u_i\|_{H^{n/2}(M)} \le CE(u_i) + C|\bar{u}_i| \le C.$$

Then there exists a subsequnce of $\{u_i\}$ weakly converges to u_∞ in $H^{\frac{n}{2}}(M)$ and strongly in $L^p(M)$ for any $1 \le p < +\infty$. Then Adams-Fontana's inequality yields that for any $q > 1$

$$\int_M e^{qu_\infty} \mathrm{d}\mu_g \le C \exp(C(q)\|u_\infty\|_{H^{\frac{n}{2}}(M)}).$$

Following the argument in Sect. 3 in [17] and the fact $|e^t - 1| \le |t|(e^t + e^{-t})$, there holds

$$\begin{aligned}
&\int_M (e^{nu_i} - e^{nu_\infty})^2 \mathrm{d}\mu_g \\
&\quad = \int_M e^{2nu_\infty}(e^{n(u_i - u_\infty)} - 1)^2 \mathrm{d}\mu_g \\
&\quad \le \int_M e^{2nu_\infty} \left(e^{n(u_i-u_\infty)} + e^{-n(u_i-u_\infty)}\right)^2 n^2 |u_i - u_\infty|^2 \mathrm{d}\mu_g \\
&\quad \le n^2 \left(\int_M e^{8nu_\infty} \mathrm{d}\mu_g\right)^{1/4} \left(\int_M \left(e^{n(u_i-u_\infty)} + e^{-n(u_i-u_\infty)}\right)^8 \mathrm{d}\mu_g\right)^{1/4} \\
&\|u_i - u_\infty\|^2_{L^4(M)} \\
&\quad \le C\|u_i - u_\infty\|^2_{L^4(M)} \to 0
\end{aligned}$$

which concludes that $u_\infty \in \mathcal{M}^*_f$. Using the theory of Lagrange multiplier, u_∞ weakly solves the following equation

$$P_g u_\infty = \lambda_1 f e^{nu_\infty} + \lambda_2 e^{nu_\infty}. \tag{2.3}$$

Integrating (2.3) over M, we get $\lambda_2 = 0$. Obviously, $\lambda_1 \neq 0$ otherwise u_∞ is a constant which contradicts to $\bar{f} < 0$. Moreover, there holds $\mathcal{A} = E(u_\infty) > 0$.

Now, we claim $\lambda_1 > 0$ and argue by contradiction. Supposing $\lambda_1 < 0$, consider

$$\psi(t) = \int_M f e^{ntu_\infty}$$

where $t \in [0, 1]$. Taking the same strategy as before and using the fact $|e^t - 1 - t| \leq t^2(e^t + e^{-t})$, for fixed t_0 and $|\epsilon| < 1$, we have

$$
\begin{aligned}
&|\psi(t_0+\epsilon) - \psi(t_0) - \epsilon n \int_M f e^{nt_0u_\infty} u_\infty \mathrm{d}\mu_g| \\
&= \left| \int_M f e^{nt_0u_\infty}(e^{n\epsilon u_\infty} - 1 - n\epsilon u_\infty)\mathrm{d}\mu_g \right| \\
&\leq \int_M |f| e^{nt_0u_\infty} \epsilon^2 n^2 u_\infty^2 (e^{n\epsilon u_\infty} + e^{-n\epsilon u_\infty})^2 \mathrm{d}\mu_g \\
&\leq C(f, n, t_0, \|u_\infty\|_{H^{\frac{n}{2}}(M)})\epsilon^2.
\end{aligned}
$$

Hence, there holds $\psi(t) \in C^1$. Meanwhile, $\psi(1) = 0$ due to $u_\infty \in \mathcal{M}_f^*$. Direct computation yields that

$$
\psi'(1) = n \int_M u_\infty f e^{nu_\infty} \mathrm{d}\mu_g = \lambda_1 \int_M u_\infty P_g u_\infty \mathrm{d}\mu_g < 0.
$$

Then there exists $t_1 \in (0, 1)$ such that $\psi(t_1) > 0$. Recall $\psi(0) = \int_M f \mathrm{d}\mu_g < 0$ and immediately there exists $t_2 \in (0, t_1)$ such that $\psi(t_2) = 0$ which means $t_2 u_\infty \in \mathcal{M}_f$. Although $t_2 u_\infty$ doesn't belong to $\mathcal{M}_f^*$, the property (2.2) shows that

$$
\mathcal{A} \leq E(t_2 u_\infty) = t_2^2 \mathcal{A} < \mathcal{A}
$$

which is impossible. Hence $\lambda_1 > 0$. Then $\tilde{u}_\infty = u_\infty + \frac{1}{n}\log \lambda_1$ satisfies

$$
P_g \tilde{u}_\infty = sgn(-\bar{f}) f e^{n\tilde{u}_\infty}.
$$

□

Now, we are going to use the above continuity trick to show the monotonicity of energy with respect to λ.

Lemma 2.2 *$\beta(\lambda)$ is strictly decreasing in $(0, -\bar{f}_0)$.*

Proof Given $0 < \lambda_1 < \lambda_2 < -\bar{f}_0$, by Theorem 2.1, we know that β_λ is achieved by some $u_\lambda \in \mathcal{M}^*(\lambda)$ for any $\lambda \in [\lambda_1, \lambda_2]$. Set

$$
\xi(t) = \int_M f_{\lambda_2} e^{n(1-t)u_{\lambda_1}} \mathrm{d}\mu_g.
$$

Notice that

$$
\xi(0) = \int_M (f_{\lambda_1} + \lambda_2 - \lambda_1) e^{nu_{\lambda_1}} \mathrm{d}\mu_g = \lambda_2 - \lambda_1 > 0,
$$

and $\xi(1) < 0$. Making use of the continuity of $\xi(t)$, there exists $t_1 \in (0, 1)$ such that $\xi(t_1) = 0$. Hence

$$\beta(\lambda_2) \leq \int_M (1-t_1)u_{\lambda_1} P_g (1-t_1)u_{\lambda_1} \mathrm{d}\mu_g = (1-t_1)^2 \beta(\lambda_1) < \beta(\lambda_1).$$

□

Applying Lebesgue's Theorem, we observe that $\beta(\lambda)$ is almost everywhere differentiable.

Lemma 2.3 *$\beta(\lambda)$ is Lipschitz continuous on compact subset of $(0, -\bar{f}_0)$ and $\beta'(\lambda) = -\frac{2\alpha_\lambda}{n}$ a.e.*

Proof Multiply f_λ to both sides of the Eq. (1.7) and integrate it over M to get

$$\alpha_\lambda = \frac{\int_M u_\lambda P_g f_\lambda \mathrm{d}\mu_g}{\int_M f_\lambda^2 e^{nu_\lambda} \mathrm{d}\mu_g} = \frac{\int_M (u_\lambda - \bar{u}_\lambda) P_g f_0 \mathrm{d}\mu_g}{\int_M f_\lambda^2 e^{nu_\lambda} \mathrm{d}\mu_g}.$$

Using Hölder's inequality, one has

$$0 < \|f_\lambda\|_{L^1}^2 \leq \left(\int_M f_\lambda^2 e^{nu_\lambda} \mathrm{d}\mu_g \right) \int_M e^{-nu_\lambda} \mathrm{d}\mu_g.$$

Then

$$\alpha_\lambda \leq C \frac{\|u_\lambda - \bar{u}_\lambda\|_{L^2} \|f_0\|_{H^n}}{\|f_\lambda\|_{L^1}^2} \int_M e^{-nu_\lambda} \mathrm{d}\mu_g.$$

As mentioned before, $\alpha_\lambda > 0$ by using Theorem 2.1.

With help of Adams-Fontana's inequality and $u_\lambda \in \mathcal{M}^*(\lambda)$, we have

$$\int_M e^{-nu_\lambda} \mathrm{d}\mu_g = \int_M e^{-n(u_\lambda - \bar{u}_\lambda)} \mathrm{d}\mu_g \int_M e^{n(u_\lambda - \bar{u}_\lambda)} \mathrm{d}\mu_g \leq e^{C_1 \int_M u_\lambda P_g u_\lambda \mathrm{d}\mu_g + C}$$

where C_1 and C are both positive constants just depending on (M, g). The positivity of P_g shows that

$$\|u_\lambda - \bar{u}_\lambda\|_{L^2(M)}^2 \leq C \int_M u_\lambda P_g u_\lambda \mathrm{d}\mu_g.$$

We could combine these estimates to get

$$\alpha_\lambda \leq C \frac{\|f_0\|_{H^n}}{\|f_\lambda\|_{L^1}^2} \sqrt{\beta(\lambda)} e^{C_1 \beta(\lambda)}.$$

If $\lambda \in [a, b] \subset (0, -\bar{f}_0)$, we just need to construct a function u such $\int_M f_a e^{nu} \mathrm{d}\mu_g > 0$ to control the upper bound β_λ with help of the trick in Lemma 2.2 since

$$\int_M f_\lambda e^{nu} \mathrm{d}\mu_g \geq \int_M f_a e^{nu} \mathrm{d}\mu_g.$$

The construction of such function is easy and we omit the details. Thus α_λ and β_λ are both uniformly bounded on compact subsets of $(0, -\bar{f}_0)$.

Consider δ close to 0 and μ close to λ with $\mu, \lambda \in [a, b]$

$$\begin{aligned}\int_M f_\mu e^{n(1+\delta)u_\lambda} \mathrm{d}\mu_g &= \int_M (f_\lambda + (\mu - \lambda)) e^{n(1+\delta)u_\lambda} \mathrm{d}\mu_g \\ &= \int_M f_\lambda e^{nu_\lambda} (e^{n\delta u_\lambda} - 1) \mathrm{d}\mu_g \\ &\quad + (\mu - \lambda) \int_M e^{nu_\lambda} (e^{n\delta u_\lambda} - 1) \mathrm{d}\mu_g + \mu - \lambda \\ &= n\delta \int_M f_\lambda e^{nu_\lambda} u_\lambda \mathrm{d}\mu_g + O(\delta^2) + (\mu - \lambda) O(\delta) + \mu - \lambda \\ &= \frac{n\beta(\lambda)}{\alpha_\lambda} \delta + O(\delta^2) + (\mu - \lambda) O(\delta) + \mu - \lambda\end{aligned}$$

where we have used $u_\lambda \in \mathcal{M}^*(\lambda)$ and the Eq. (1.7). Hence we can choose

$$\delta = -\frac{\alpha_\lambda}{n\beta(\lambda)} (\mu - \lambda) + O((\mu - \lambda)^2)$$

such that

$$\int_M f_\mu e^{n(1+\delta)u_\lambda} \mathrm{d}\mu_{g_b} = 0.$$

Then

$$\beta(\mu) \le (1 + \delta)^2 \beta(\lambda) = \beta(\lambda) - \frac{2\alpha_\lambda}{n} (\mu - \lambda) + O((\mu - \lambda)^2).$$

Similarly, we have

$$\beta(\lambda) \le \beta(\mu) - \frac{2\alpha_\mu}{n} (\lambda - \mu) + O((\mu - \lambda)^2).$$

Hence

$$-\frac{2\alpha_\mu}{n} (\mu - \lambda) + O((\mu - \lambda)^2) \le \beta(\mu) - \beta(\lambda) \le -\frac{2\alpha_\lambda}{n} (\mu - \lambda) + O((\mu - \lambda)^2). \tag{2.4}$$

Due to the estimate (2.4), we show that $\beta(\lambda)$ is Lipschitz continuous on compact subset of $(0, -\bar{f}_0)$. Moreover, we have

$$\frac{\mathrm{d}\beta}{\mathrm{d}\lambda}(\lambda_+) \le -\frac{2\alpha_\lambda}{n}, \quad \frac{\mathrm{d}\beta}{\mathrm{d}\lambda}(\lambda_-) \ge -\frac{2\alpha_\lambda}{n}.$$

Since $\beta(\lambda)$ is differentiable almost everywhere, one has

$$\beta'(\lambda) = -\frac{2\alpha_\lambda}{n} \quad a.e.$$

□

Lemma 2.4 *There hold*

$$\liminf_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log(1/\lambda)}\geq\frac{(n-1)!|\mathbb{S}^n|}{n},\quad \limsup_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log(1/\lambda)}\leq (n-1)!|\mathbb{S}^n|.$$

Moreover, if there exists a maximum point of $f_0(x)$ is flat up to $n-1$ order, one has

$$\limsup_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log(1/\lambda)}\leq\frac{2}{n}(n-1)!|\mathbb{S}^n|,\quad \liminf_{\lambda\downarrow 0}\lambda\alpha_\lambda\leq (n-1)!|\mathbb{S}^n|.$$

Proof Firstly, following the argument of Lemma 4.1 in [26], Hölder's inequality together with Adams-Fontana inequality (see Proposition 2.2 in [25]) shows that

$$\begin{aligned}
0<\|f_0\|^2_{L^1(M)} &\leq \int_M |f_0|e^{-nu_\lambda}\mathrm{d}\mu_g\int_M |f_0|e^{nu_\lambda}\mathrm{d}\mu_g\\
&=\int_M |f_0|e^{-nu_\lambda}\mathrm{d}\mu_g\int_M \lambda e^{nu_\lambda}\mathrm{d}\mu_g\\
&\leq C\lambda\int_M e^{-nu_\lambda}\mathrm{d}\mu_g\int_M e^{nu_\lambda}\mathrm{d}\mu_g\\
&= C\lambda\int_M e^{n(-u_\lambda+\bar{u}_\lambda)}\mathrm{d}\mu_g\int_M e^{n(u_\lambda-\bar{u}_\lambda)}\mathrm{d}\mu_g\\
&\leq C\lambda\exp\left(\frac{n}{(n-1)!|\mathbb{S}^n|}\beta(\lambda)\right).
\end{aligned}$$

Then

$$\beta(\lambda)-\frac{(n-1)!|\mathbb{S}^n|}{n}\log\frac{1}{\lambda}\geq C. \tag{2.5}$$

Immediately, there holds

$$\liminf_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log 1/\lambda}\geq\frac{(n-1)!|\mathbb{S}^n|}{n}.$$

Of course, we also have

$$\beta(\lambda)\to\infty,\quad \text{as}\quad \lambda\to 0. \tag{2.6}$$

As for the upper bound for $\beta(\lambda)$, consider a test function introduced in [11] (see Sect. 4 of [11]). For $\delta>0$ small to be chosen later, consider a non-decreasing smooth cut-off function $\chi_\delta(t)$ such that $\chi_\delta(t)=t$ for $t\in[0,\delta]$ and $\chi_\delta(t)=2\delta$ for any $t\geq 2\delta$. For $s>0$ to be chosen later, define the test function $\varphi_{s,\delta}$ as

$$\varphi_{s,\delta}(x)=\log\frac{2s}{1+s^2\chi_\delta(d_g(x,x_0))^2} \tag{2.7}$$

where x_0 is a maximum point of f_0 and $d_g(x, y)$ is the distance function on (M, g). Letting $s \to \infty$, Lemma 4.2 in [11] and Lemma 4.5 in [25] show that

$$\int_M \varphi_{s,\delta} P_g \varphi_{s,\delta} \mathrm{d}\mu_g \leq (2(n-1)!|\mathbb{S}^n| + o_\delta(1)) \log s + C_\delta. \tag{2.8}$$

Due to $f_0(x_0) = \max_M f_0 = 0$, we could find $c_1 > 0$ and small $\delta > 0$ such that

$$f_0(x) \geq -c_1 d_g(x, x_0)^2$$

for any $x \in B_{2\delta}(x_0)$. Thus for such small $\delta > 0$ fixed

$$\begin{aligned}
&\int_M f_\lambda e^{n\varphi_{s,\delta}} \mathrm{d}\mu_g \\
&= \int_{B_{2\delta}(x_0)} (f_0 + \lambda) \left(\frac{2s}{1 + s^2 \chi_\delta (d_g(x, x_0))^2} \right)^n \mathrm{d}\mu_g \\
&\quad + \int_{M \backslash B_{2\delta}(x_0)} (f_0 + \lambda)) \left(\frac{2s}{1 + 4s^2\delta^2} \right)^n \mathrm{d}\mu_g \\
&\geq \int_{B_\delta(x_0)} \lambda \left(\frac{2s}{1 + s^2 d_g(x, x_0)^2} \right)^n \mathrm{d}\mu_g - c_1 \\
&\quad \int_{B_\delta(x_0)} d_g(x, x_0)^2 \left(\frac{2s}{1 + s^2 d_g(x, x_0)^2} \right)^n \mathrm{d}\mu_g \\
&\quad - c_1 \int_{B_{2\delta}(x_0) \backslash B_\delta(x_0)} d_g(x, x_0)^2 \left(\frac{2s}{1 + s^2\delta^2} \right)^n \mathrm{d}\mu_g + O(s^{-n}) \\
&\geq C_1 \lambda \int_{|x| \leq \delta} \frac{s^n}{(a_1 + s^2|x|^2)^n} \mathrm{d}x - C_2 \int_{|x| \leq \delta} \frac{s^n |x|^2}{(a_2 + s^2|x|^2)^n} \mathrm{d}x + O(s^{-n})
\end{aligned}$$

where C_1, C_2, a_1 and a_2 are all positive constants independent of s by considering the exponential map in the small neighborhood. Now using polar coordinates, we have

$$\begin{aligned}
&\int_M f_\lambda e^{n\varphi_{s,\delta}} \mathrm{d}\mu_g \\
&\geq C_3 \lambda \int_0^{s\delta} \frac{t^{n-1}}{(a_1 + t^2)^n} \mathrm{d}t - C_4 s^{-2} \int_0^{s\delta} \frac{t^{n+1}}{(a_2 + t^2)^n} \mathrm{d}t + O(s^{-n}) \\
&= C_3 \lambda \int_0^{\infty} \frac{t^{n-1}}{(a_1 + t^2)^n} \mathrm{d}t - C_4 s^{-2} \int_0^{\infty} \frac{t^{n+1}}{(a_2 + t^2)^n} \mathrm{d}t + O(s^{-n}).
\end{aligned}$$

We choose

$$s^{-2} = c\lambda \tag{2.9}$$

where c is a positive constant such that

$$\int_M (f_0+\lambda)e^{n\varphi_{s,\delta}}\mathrm{d}\mu_g \geq Cs^{-2} + O(s^{-n}).$$

Hence if λ is small enough, we have

$$\int_M (f_0+\lambda)e^{n\varphi_{s,\delta}}\mathrm{d}\mu_g > 0.$$

Due to $\bar{f}_\lambda < 0$, the same trick in Lemma 2.2 implies that there exists $t_1 \in (0,1)$ such that $t_1\varphi_{s,\delta} \in \mathcal{M}(\lambda)$. Thus with help of (2.9) and (2.8), there holds

$$\beta(\lambda) \leq E(t_1\varphi_{s,\delta}) \leq ((n-1)!|\mathbb{S}^n| + o_\delta(1))\log\frac{1}{\lambda} + C(\delta).$$

Immediately, one has

$$\limsup_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log 1/\lambda} \leq (n-1)!|\mathbb{S}^n| + o_\delta(1).$$

Then by letting $\delta \to 0$, we obtain the desired result.

Now, if a maximum point x_0 of f_0 is flat up to $n-1$ order, we have

$$f_0(x) \geq -Cd_g(x,x_0)^n$$

in the neighborhood $B_{2\delta}(x_0)$. Similarly, a direct computation yields that

$$\begin{aligned}&\int_M (f_0+\lambda)e^{n\varphi_{s,\delta}}\mathrm{d}\mu_g\\ &\quad\geq C\lambda\int_0^{s\delta}\frac{t^{n-1}}{(a_1+t^2)^n}\mathrm{d}t - Cs^{-n}\int_0^{s\delta}\frac{t^{2n-1}}{(a_2+t^2)^n}\mathrm{d}t + O(s^{-n}). \end{aligned}\tag{2.10}$$

Now for any small $\epsilon > 0$, we choose

$$s^{-n+\epsilon} = \lambda$$

such that

$$\int_M (f_0+\lambda)e^{n\varphi_{s,\delta}}\mathrm{d}\mu_g \geq Cs^{-n+\epsilon} + O(s^{-n}\log s).$$

Hence if λ small enough, we have

$$\int_M (f_0+\lambda)e^{n\varphi_{s,\delta}}\mathrm{d}\mu_g > 0.$$

Samely as before, one concludes that

$$\limsup_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log 1/\lambda}\le\frac{2}{n-\epsilon}(n-1)!|\mathbb{S}^n|+o_\delta(1).$$

Letting $\delta\to 0$ and then $\epsilon\to 0$, one has

$$\limsup_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log 1/\lambda}\le\frac{2}{n}(n-1)!|\mathbb{S}^n|. \tag{2.11}$$

Meanwhile, with help of Lemma 2.3 and the above estimate (2.11), we have

$$\liminf_{\lambda\downarrow 0}\lambda\alpha_\lambda\le\frac{n}{2}\liminf_{\lambda\downarrow 0}\lambda|\frac{\mathrm{d}\beta_\lambda}{\mathrm{d}\lambda}|\le\frac{n}{2}\limsup_{\lambda\downarrow 0}\frac{\beta(\lambda)}{\log 1/\lambda}\le(n-1)!|\mathbb{S}^n|. \tag{2.12}$$

Indeed, suppose $\lambda_1>0$ and $c_0>\lim_{\lambda\downarrow 0}\sup\frac{\beta(\lambda)}{\log 1/\lambda}$ for almost $0<\lambda<\lambda_1$ such that $|\beta'_\lambda|\ge\frac{c_0}{\lambda}$ then for any sufficiently small λ we obtain

$$\beta_\lambda-\beta_{\lambda_1}\ge\int_\lambda^{\lambda_1}|\beta'_s|\mathrm{d}s\ge c_0\int_\lambda^{\lambda_1}\frac{\mathrm{d}s}{s}\ge c_0\log(1/\lambda)+C$$

which contradicts to our choice c_0 by letting $\lambda\downarrow 0$. Finally, we complete our proof. □

3 Blow-Up Analysis

The following lemma modifies Proposition 3.1 of [23] by Malchiodi for future application.

Lemma 3.1 *Let (M,g) be a compact closed manifold with even dimension $n\ge 4$ and $\ker P_g=\{constants\}$. Given a sequence $\{u_k\}$ satisfying $P_g u_k=h_k$ with $h_k\in L^1(M)$ and $\int_M h_k^+\mathrm{d}\mu_g\le C$ where C is a constant independent of k. Set $\bar{u}_k=\frac{1}{vol(M,g)}\int_M u_k\mathrm{d}\mu_g$. Then one of the following is true:*

(a) *There exists a constant $q_0>n$ such that*

$$\int_M e^{q_0(u_k-\bar{u}_k)}\mathrm{d}\mu_g\le C.$$

Moreover, if $h_k^+\le Ce^{nu_k}$, there holds

$$\|(u_k-\bar{u}_k)^+\|_{L^\infty(M)}\le C.$$

(b) *There exist finite points p_i with $1\le i\le i_0$ such that for any $s>0$*

$$\liminf_{k\to\infty}\int_{B_s(p_i)}h_k^+\mathrm{d}\mu_g\ge\frac{1}{2}(n-1)!|\mathbb{S}^n|. \tag{3.1}$$

Moreover, if $h_k^+ \le Ce^{nu_k}$, *on any* $B_r(x) \subset\subset M\backslash\{p_1, \cdots, p_{i_0}\}$, *there holds*

$$\|(u_k - \bar{u}_k)^+\|_{L^\infty(B_r(x))} \le C.$$

Proof Due to $\ker P_g = \{constants\}$, there exists a Green's function (see [8, 23, 25]) satisfying

$$P_g G(x, y) = \delta_y(x) - 1$$

with

$$\left| G(x, y) - \frac{2}{(n-1)!|\mathbb{S}^n|} \log \frac{1}{d_g(x, y)} \right| \le C$$

near the diagonal of $M \times M$. Since M is compact and $P_g(u + C) = P_g u$ for any constant C, we could choose a positive Green's function. From now on, we assume that $G(x, y) > 0$. Based on our assumptions, there holds

$$\int_M h_k^+ \mathrm{d}\mu_g \le C. \tag{3.2}$$

For any $x_0 \in M$, suppose that there exists $r_0 > 0$ and $a_0 > 0$ such that

$$\int_{B_{3r_0}(x_0)} h_k^+ \mathrm{d}\mu_g \le \frac{1}{2}(n-1)!|\mathbb{S}^n| - a_0 \tag{3.3}$$

for sufficiently large k. Then, for any $x \in B_{r_0}(x_0)$, we have

$$\begin{aligned}
u_k(x) - \bar{u}_k &= \int_M G(x, y) h_k \mathrm{d}\mu_g \\
&\le \int_M G(x, y) h_k^+ \mathrm{d}\mu_g \\
&= \int_{B_{2r_0}(x_0)} G(x, y) h_k^+(y) \mathrm{d}\mu_g(y) \\
&\quad + \int_{M\backslash B_{2r_0}(x_0)} G(x, y) h_k^+(y) \mathrm{d}\mu_g(y) \\
&\le \int_{B_{2r_0}(x_0)} G(x, y) h_k^+ \mathrm{d}\mu_{g_b}(y) + C
\end{aligned}$$

since $|G(x, y)| \le C$ on $B_{r_0}(x_0) \times (M\backslash B_{2r_0}(x_0))$ and the estimate (3.2). Choose a cut-off function $0 \le \eta \le 1$ with $\eta \equiv 1$ in $B_{2r_0}(x_0)$ and η vanishes outside $B_{3r_0}(x_0)$.

For $\alpha > 0$, take the strategy used in [5] and make use of Jensen's inequality to get

$$\begin{aligned}
&\int_{B_{r_0}(x_0)} e^{\alpha(u_k(x)-\bar{u}_k)}\mathrm{d}\mu_g \\
&\leq C\int_{B_{r_0}(x_0)} \exp\left(\int_{B_{2r_0}(x_0)} \alpha G(x,y)h_k^+(y)\mathrm{d}\mu_g(y)\right)\mathrm{d}\mu_g(x) \\
&\leq C\int_{B_{r_0}(x_0)} \exp\left(\int_M \alpha G(x,y)h_k^+(y)\eta(y)\mathrm{d}\mu_g(y)\right)\mathrm{d}\mu_g(x) \\
&\leq C\int_{B_{r_0}(x_0)}\int_M \exp\left(\alpha\|h_k^+\eta\|_{L^1(M)}G(x,y)\right)\frac{h_k^+\eta}{\|h_k^+\eta\|_{L^1(M)}}\mathrm{d}\mu_g(y)\mathrm{d}\mu_g(x) \\
&\leq C\int_M \frac{h_k^+\eta}{\|h_k^+\eta\|_{L^1(M)}}\int_M\left(\frac{1}{d_g(x,y)}\right)^{\frac{2\alpha\|h_k^+\eta\|_{L^1(M)}}{(n-1)!|\mathbb{S}^n|}}\mathrm{d}\mu_g(x)\mathrm{d}\mu_g(y).
\end{aligned}$$

With help of (3.3), there exists $\alpha_0 > n$ such that the last integral is finite i.e.

$$\int_{B_{r_0}(x_0)} e^{\alpha_0(u_k-\bar{u}_k)}\mathrm{d}\mu_g \leq C. \tag{3.4}$$

If any $x \in M$, the estimate (3.3) holds. Since M is compact, there exist finite balls covering M and then there exists $q_0 > n$ such that

$$\int_M e^{q_0(u_k-\bar{u}_k)}\mathrm{d}\mu_g \leq C.$$

Otherwise, due to $\int_M h_k^+\mathrm{d}\mu_g \leq C$, there are finitely many points such (3.1) holds.

Now, if the condition $h_k^+ \leq Ce^{nu_k}$ satisfies in addition, for each x_0 as before such (3.3) holds. Then for any $x \in B_{r_0/2}(x_0)$, with help of (3.4), one has

$$\begin{aligned}
u_k(x)-\bar{u}_k &\leq C + C\int_{B_{r_0}(x_0)} G(x,y)e^{nu_k}\mathrm{d}\mu_g(y) \\
&\leq C + C\left(\int_{B_{r_0}(x_0)} e^{\alpha_0 u_k}\mathrm{d}\mu_g\right)^{n/\alpha_0}\left(\int_{B_{r_0}(x_0)} |G|^{\frac{1}{1-n/\alpha_0}}\mathrm{d}\mu_g\right)^{1-n/\alpha_0} \\
&\leq C
\end{aligned}$$

which concludes that

$$\|(u_k-\bar{u}_k)^+\|_{L^\infty(B_{r_0/2}(x_0))} \leq C. \tag{3.5}$$

If for any $x \in M$, the estimate (3.3) holds. With help of a finite covering, one has

$$\|(u_k-\bar{u}_k)^+\|_{L^\infty(M)} \leq C. \tag{3.6}$$

Finally, we finish our proof.

□

From now on, we assume f_0 has a l-type maximum point with $l \geq n$. Based on the definition, it is not hard to see that such l-type maximum point is flat up to $n-1$ order. Meanwhile, we have

$$|f_\lambda| \leq \lambda - f_0 = 2\lambda - f_\lambda, \quad f_\lambda^+ \leq \lambda$$

and then the estimate (2.12) yields that

$$\liminf_{\lambda \downarrow 0} \int_M \alpha_\lambda |f_\lambda| e^{nu_\lambda} \mathrm{d}\mu_g \leq 2(n-1)!|\mathbb{S}^n|, \tag{3.7}$$

$$\liminf_{\lambda \downarrow 0} \int_M \alpha_\lambda f_\lambda^+ e^{nu_\lambda} \mathrm{d}\mu_g \leq (n-1)!|\mathbb{S}^n|. \tag{3.8}$$

Based on (2.12), we choose a subsequence $\lambda_k \to 0$ such that

$$\lim_{k \to \infty} \lambda_k \alpha_{\lambda_k} \leq (n-1)!|\mathbb{S}^n|. \tag{3.9}$$

For simplicity, set $u_k = u_{\lambda_k}$, $g_k = e^{2u_k} g$ and $Q_k := \alpha_{\lambda_k} f_{\lambda_k} \leq \alpha_{\lambda_k} \lambda_k$. Recalling the Eq. (1.7), there holds

$$P_g u_k = Q_k e^{nu_k}.$$

Lemma 3.2 *Consider the sequence $\{u_k\}$ as above satisfying (3.9).*

(i) *As $k \to \infty$, there holds*

$$\bar{u}_k := \frac{1}{vol(M,g)} \int_M u_k \mathrm{d}\mu_g \to -\infty, \quad \alpha_{\lambda_k} \to \infty.$$

(ii) *There exist i_0 points p_i with $1 \leq i \leq i_0$, where $i_0 \in \{1, 2\}$, such that $f_0(p_i) = 0$ for all i, and for any $s > 0$ fixed,*

$$\liminf_{k \to \infty} \int_{B_s(p_i)} Q_k^+ e^{nu_k} \mathrm{d}\mu_g \geq \frac{1}{2}(n-1)!|\mathbb{S}^n| \tag{3.10}$$

as well as

$$\liminf_{k \to \infty} \int_{B_s(p_i)} e^{nu_k} \mathrm{d}\mu_g \geq \frac{1}{2}, \quad \liminf_{k \to \infty} \lambda_k \alpha_{\lambda_k} \geq \frac{1}{2}(n-1)!|\mathbb{S}^n|.$$

Proof Firstly, we claim that

$$\bar{u}_k \to -\infty, \quad \text{as} \quad k \to \infty. \tag{3.11}$$

Based on $u_k \in \mathcal{M}^*(\lambda_k)$, there holds

$$\int_M (-f_0) e^{nu_k} \mathrm{d}\mu_g = \lambda_k \int_M e^{nu_k} \mathrm{d}\mu_g = \lambda_k$$

and apply Jensen's inequality to get

$$\exp\left(\int_M nu_k \frac{(-f_0)}{\|f_0\|_{L^1(M)}} \mathrm{d}\mu_g\right) \leq \int_M e^{nu_k} \frac{-f_0}{\|f_0\|_{L^1(M)}} \mathrm{d}\mu_g = \frac{\lambda_k}{\|f_0\|_{L^1(M)}}.$$

Hence we have

$$\int_M (-f_0) u_k \mathrm{d}\mu_g \to -\infty, \quad \text{as} \quad k \to \infty.$$

Set the notations $\varphi^+ := \max\{0, \varphi\}$ and $\varphi^- := \max\{0, -\varphi\}$ for any function φ. Notice that

$$\begin{aligned} &|\int_M (-f_0) u_k^- \mathrm{d}\mu_g + \int_M (-f_0) u_k \mathrm{d}\mu_g| = \int_M (-f_0) u_k^+ \mathrm{d}\mu_g \\ &\leq \frac{\max_M (-f_0)}{n} \int_M e^{nu_k} \mathrm{d}\mu_g \leq C, \end{aligned}$$

and

$$\int_M (-f_0) u_k^- \mathrm{d}\mu_g \leq \max_M (-f_0) \int_M u_k^- \mathrm{d}\mu_g$$

which yields

$$\int_M u_k^- \mathrm{d}\mu_g \to +\infty, \quad \text{as} \quad k \to \infty.$$

Then with help of the fact $x \leq e^x$,

$$\begin{aligned} \int_M u_k \mathrm{d}\mu_g &= \int_M u_k^+ \mathrm{d}\mu_g - \int_M u_k^- \mathrm{d}\mu_g \\ &\leq \frac{1}{n} \int_M e^{nu_k} \mathrm{d}\mu_g - \int_M u_k^- \mathrm{d}\mu_g \\ &= \frac{1}{n} - \int_M u_k^- \mathrm{d}\mu_g \to -\infty. \end{aligned}$$

Thus we prove our claim (3.11).

With help of (3.9), one has

$$\int_M Q_k^+ e^{nu_k} \mathrm{d}\mu_g \leq \lambda_k \alpha_{\lambda_k} \int_M e^{nu_k} \mathrm{d}\mu_g \leq C, \quad Q_k^+ e^{nu_k} \leq C e^{nu_k}.$$

We will apply Lemma 3.1 and rule out Case (a) in Lemma 3.1. We argue by contradiction. Supposing that Case (a) in Lemma 3.1 holds, there holds

$$\|(u_k - \bar{u}_k)^+\|_{L^\infty(M)} \leq C$$

and then

$$1=\int_M e^{nu_k}\mathrm{d}\mu_g=\int_M e^{n(u_k-\bar{u}_k)}\mathrm{d}\mu_g e^{n\bar{u}_k}\leq Ce^{n\bar{u}_k}. \tag{3.12}$$

which contradicts to (3.11). Thus there are only finite points $p_i\in M$ with $1\leq i\leq i_0$, $i_0\in\mathbb{N}$ such (3.10) holds. With help of the estimate

$$\int_{B_s(p_i)} Q_k^+ e^{nu_k}\mathrm{d}\mu_g\leq \alpha_{\lambda_k}\lambda_k\int_M e^{nu_k}\mathrm{d}\mu_g=\alpha_{\lambda_k}\lambda_k,$$

there holds

$$\liminf_{k\to\infty}\alpha_{\lambda_k}\lambda_k\geq\frac{1}{2}(n-1)!|\mathbb{S}^n|.$$

Immediately, one has $\alpha_{\lambda_k}\to\infty$ as $\lambda_k\to 0$. Meanwhile, Due to $Q_k^+\leq\alpha_{\lambda_k}\lambda_k$, the estimates (3.9) and (3.10) yield that

$$\liminf_{k\to\infty}\int_{B_s(p_i)} e^{nu_k}\mathrm{d}\mu_g\geq\frac{1}{2}. \tag{3.13}$$

Based on the unit volume property $\int_M e^{nu_k}\mathrm{d}\mu_g=1$, i_0 is either 1 or 2.

The remaining task is to demonstrate $f(p_i)=0$ for each $1\leq i\leq i_0$. We can establish this argument by means of a contradiction. Suppose that $f_0(p_i)<0$ for some i. Then, we arrive at a contradiction because there exists $s>0$ such that for any $x\in B_s(p_i)$ and sufficiently large k

$$Q_k=\alpha_{\lambda_k}(\lambda_k+f_0(x))\leq 0,$$

and then

$$\int_{B_s(p_i)} Q_k^+ e^{nu_k}\mathrm{d}\mu_g\leq 0$$

which contradicts to (3.10).

In conclusion, we have successfully demonstrated all desired results. Thus, the proof is complete. □

Remark 3.3 Just as Remark 3.2 in [23], using the same proof, we can extend Lemma 3.2 to the case in which also the metric on M depends on k, and converges to some smooth g in $C^m(M)$ for any integer m. We will use this variant later.

Following the argument of Lemma 2.3 in [23], it is not hard to get the following lemma with help of (3.7) and the representation of Green's function.

Lemma 3.4 *There is a constant depending only on M, f_0, j and p such that for any $1\leq j\leq n-1$ and $p>0$ satisfying $jp<n$, small $r>0$ and any $x\in M$, there holds*

$$\int_{B_r(x)}|\nabla^j u_k|^p\mathrm{d}\mu_g\leq Cr^{n-jp}.$$

Proof The proof is the same as Lemma 2.3 of [23]. We omit the details here. □

With help of Lemma 3.2, there exists a small constant $\delta > 0$ such that $B_{2\delta}(p_i)$ are disjoint. For any blow-up point p_i (we denote it as P for simplicity) and $0 < r < \delta$, there exists $x_{r,k} \in \bar{B}_\delta(P)$ such that

$$\int_{B_r(x_{r,k})} e^{nu_k} \mathrm{d}\mu_g = \sup_{x \in \bar{B}_\delta(P)} \int_{B_r(x)} e^{nu_k} \mathrm{d}\mu_g.$$

Then for each fixed radius r, using Lemma 3.2, there holds

$$\lim_{k\to\infty} \inf \int_{B_r(x_{r,k})} e^{nu_k} \mathrm{d}\mu_g \geq \lim_{k\to\infty} \inf \int_{B_r(P)} e^{nu_k} \mathrm{d} mu_g \geq \frac{1}{2}.$$

Thus, if k is large enough, there exists r_k such that $\int_{B_{r_k}(x_{r_k,k})} e^{nu_k} \mathrm{d}\mu_g = \frac{1}{8}$. For simplicity, we denote $x_{r_k,k}$ as x_k and there holds

$$\int_{B_{r_k}(x_k)} e^{nu_k} \mathrm{d}\mu_g = \sup_{x \in \bar{B}_\delta(P)} \int_{B_{r_k}(x)} e^{nu_k} \mathrm{d}\mu_g = \frac{1}{8}. \tag{3.14}$$

Lemma 3.5 *There holds*

$$r_k \to 0, \quad x_k \to P.$$

Moreover, if P is a l-type maximum point of f_0, one has

$$r_k^l \leq C\lambda_k, \quad d_g(x_k, P)^l \leq C\lambda_k.$$

Proof Firstly, based on the choices of x_i and r_i, we have

$$\frac{1}{8} = \int_{B_{r_k}(x_k)} e^{nu_k} \mathrm{d}\mu_g \geq \int_{B_{r_k}(P)} e^{nu_k} \mathrm{d}\mu_g.$$

Then the estimate (3.13) yields that $r_k \to 0$. With help of $r_k \to 0$, if there is a subsequence $x_k \to x_0 \in B_\delta(P)\backslash\{P\}$, for sufficiently large k, one has

$$B_{r_k}(x_k) \subset\subset M\backslash\{p_i; 1 \leq i \leq i_0\}.$$

Then part (b) of Lemma 3.1 yields that

$$\|(u_k - \bar{u}_k)^+\|_{L^\infty(B_{r_k}(x_k))} \leq C.$$

Making use of this estimate, there holds

$$\int_{B_{r_k}(x_k)} e^{nu_k} \mathrm{d}\mu_g \leq \int_{B_{r_k}(x_k)} e^{n(u_k - \bar{u}_k)} \mathrm{d}\mu_g e^{n\bar{u}_k} \leq C e^{n\bar{u}_k} \to 0$$

as $k \to \infty$ which contradicts to our choice (3.14).

Based on the Definition 1.1, there exists a constant $C_1 > 0$ such that

$$supp(Q_k^+) \cap B_\delta(P) \subset B_{C_1 \lambda_k^{\frac{1}{l}}}(P).$$

We claim that $r_k^l \lambda_k^{-1} \leq C$ and argue by contradiction. Suppose that there exist a subsequence such that $\frac{r_k^l}{\lambda_k} \to \infty$. Then, for sufficiently large k, there holds

$$B_{C_1 \lambda_k^{\frac{1}{l}}}(P) \subset B_{r_k}(P).$$

Then, with help of (3.9) and (3.14), for sufficiently large k, there holds

$$\begin{aligned}
\frac{1}{4}(n-1)!|\mathbb{S}^n| &\geq \alpha_{\lambda_k} \lambda_k \int_{B_{r_k}(x_k)} e^{nu_k} \mathrm{d}\mu_g \\
&\geq \alpha_{\lambda_k} \lambda_k \int_{B_{r_k}(P)} e^{nu_k} \mathrm{d}\mu_g \\
&\geq \int_{B_{r_k}(P)} Q_k^+ e^{nu_k} \mathrm{d}\mu_g \\
&\geq \int_{B_{C_1 \lambda_k^{1/l}}(P)} Q_k^+ e^{nu_k} \mathrm{d}\mu_g \\
&\geq \int_{B_\delta(P)} Q_k^+ e^{nu_k} \mathrm{d}\mu_g
\end{aligned}$$

which contradicts to Lemma 3.2. Thus we prove the claim

$$r_k^l \leq C\lambda_k. \tag{3.15}$$

Due to P is a l-type maximum point of f_0, there exists $c_1 > 0$ such that for small $\delta > 0$

$$-f_0(x) \geq c_1 d_g(x, P)^l, \quad x \in B_\delta(P).$$

Using the estimate (3.9) and Lemma 3.2, there holds, for large k

$$C^{-1} \leq \lambda_k \alpha_{\lambda_k} \leq C.$$

With help of this fact and the estimate $|f_\lambda| \leq \lambda - f_0 = 2\lambda - f_\lambda$, then there holds

$$\begin{aligned}
C &\geq \int_M (2\lambda_k \alpha_{\lambda_k}) e^{nu_k} \mathrm{d}\mu_g \\
&= \int_M \alpha_{\lambda_k} (2\lambda_k - f_{\lambda_k}) e^{nu_k} \mathrm{d}\mu_g \\
&\geq \int_M \alpha_{\lambda_k} |f_{\lambda_k}| e^{nu_k} \mathrm{d}\mu_g \\
&\geq \int_{B_{r_k}(x_k)} |Q_k| e^{nu_k} \mathrm{d}\mu_g \\
&\geq \int_{B_{r_k}(x_k)} \alpha_{\lambda_k} \left(-f_0 - \lambda_k\right) e^{nu_k} \mathrm{d}\mu_g \\
&\geq c_1 \int_{B_{r_k}(x_k)} \alpha_{\lambda_k} d_g(x, P)^l e^{2u_k} \mathrm{d}\mu_g - C \\
&\geq C \int_{B_{r_k}(x_k)} \lambda_k^{-1} d_g(x, P)^l e^{2u_k} \mathrm{d}\mu_g - C
\end{aligned}$$

which deduces that

$$\int_{B_{r_k}(x_k)} \lambda_k^{-1} d_g(x, P)^l e^{2u_k} \mathrm{d}\mu_g \leq C. \tag{3.16}$$

For any $x \in B_{r_k}(x_k)$, with help of (3.15), one has

$$\lambda_k^{-\frac{1}{l}} d_g(x, P) \geq \lambda_k^{-\frac{1}{l}} d_g(x_k, P) - \lambda_k^{-\frac{1}{l}} r_k \geq \lambda_k^{-\frac{1}{l}} d_g(x_k, P) - C. \tag{3.17}$$

Combing these estimates (3.14), (3.16) and (3.17), we must have

$$\lambda_k^{-\frac{1}{l}} d_g(x_k, P) \leq C.$$

Finally, we complete our proof. □

4 Proof of Main Theorem

During doing blow-up analysis, we need to study the following conformally invariant equation:

$$(-\Delta)^{\frac{n}{2}} u(x) = Q(x) e^{nu(x)} \quad \text{on } \mathbb{R}^n \tag{4.1}$$

where $n \geq 2$ is an even integer and $Q(x)$ is a smooth function. Supposing that $Qe^{nu} \in L^1(\mathbb{R}^n)$, we say $u(x)$ is a normal solution to (4.1) if $u(x)$ satisfies the integral equation

$$u(x)=\frac{2}{(n-1)!|\mathbb{S}^n|}\int_{\mathbb{R}^n}\log\frac{|y|}{|x-y|}Q(y)e^{nu(y)}\mathrm{d}y+C \tag{4.2}$$

for some constant C. More details about normal solutions can be found in Sect. 2 of [19]. Here, we assume that $u(x)$ and $Q(x)$ are both smooth functions on $\mathbb{R}^n$. Although similar results can also be obtained under weaker regularity assumptions, for the sake of brevity, we will focus on the smooth case throughout this paper.

With help of higher order Bol's inequality established in [20], we are going to give the proof of Theorem 1.4. For the reader's convenience, we rewrite Theorems 5.2 and 5.3 as the following lemma.

Lemma 4.1 (Theorems 5.2, 5.3 in [20]) *Given a homogeneous polynomial p_k satisfying $p_k(x)\le 0$ and $x\cdot\nabla p_k(x)=kp_k(x)$ on $\mathbb{R}^n$ where $n\ge 4$ is an even integer and the integer k satisfies $2\le k\le n-2$. Suppose that $u(x)$ is the normal solution to the following equation*

$$(-\Delta)^{\frac{n}{2}}u(x)=((n-1)!+p_k(x))\,e^{nu}.$$

Then there holds

$$\int_{\mathbb{R}^n}e^{nu}\mathrm{d}x>|\mathbb{S}^n|.$$

For $k\ge n$, a Liouville type theorem is established in [16, 18, 30], and [20]. For later use, we write the following lemma which is a special case of Theorem 4.5 in [20].

Lemma 4.2 (Theorem 4.5 in [20]) *Given a homogeneous polynomial p_k satisfying $p_k(x)\le 0$ and $x\cdot\nabla p_k(x)=kp_k(x)$ on $\mathbb{R}^n$ where $n\ge 4$ is an even integer and the integer k satisfies $k\ge n$. Then there is no normal solution to the following equation*

$$(-\Delta)^{\frac{n}{2}}u(x)=((n-1)!+p_k(x))\,e^{nu}.$$

Proof of Theorem 1.4 Following the strategy taken in [23], given small $\delta>0$, consider the exponential map

$$\exp_P:\hat{B}_\delta(0)\to M,\quad \exp_P(0)=P$$

where $\hat{B}_\delta(0):=\{z\in\mathbb{R}^n||z|<\delta\}$. We define the metric on $\hat{B}_\delta(0)$ by $\tilde{g}_k:=(\exp_P)^*g$ and set

$$\tilde{u}_k=u_k\circ\exp_P,\quad z_k=\exp_P^{-1}(x_k)$$

where x_k and r_k come from (3.14). Consider the linear transformation $T_k:z\to r_kz+z_k$ and set

$$\hat{u}_k(z)=\tilde{u}_k(T_kz)+\log r_k$$

for $z\in D_k:=\{z\in\mathbb{R}^n||z_k+r_kz|<\delta\}$. Since $r_k\to 0$ and $z_k\to 0$ (see Lemma 3.5), we find that D_k exhaust $\mathbb{R}^n$ as $k\to\infty$ and $\hat{B}_{\frac{\delta}{2r_k}}(0)\subset D_k$ for sufficiently large k. Set

$\hat{g}_k = r_k^{-2} T_k^* \tilde{g}_k$ and $\hat{f}_k(z) = \lambda_k + f_0(\exp_P(r_k z + z_k))$. Due to the conformal property of P_g, there holds

$$P_{\hat{g}_k} \hat{u}_k(z) = \alpha_{\lambda_k} \hat{f}_k e^{n\hat{u}_k}, \quad z \in D_k.$$

Notice that $\hat{g}_k$ converges in $C^m_{loc}(\mathbb{R}^n)$ to the flat metric $(dz)^2$ for any integer m. Based on (3.14), using a change of variables, we have

$$\frac{1}{8} = \int_{B_{r_k}(x_k)} e^{nu_k} \mathrm{d}\mu_g = \int_{\hat{B}_1(0)} e^{n\hat{u}_k} \mathrm{d}\mu_{\hat{g}_k}.$$

Moreover, for any $y \in \hat{B}_{\frac{\delta}{2r_k}}(0)$, there holds

$$\int_{\hat{B}_{\frac{1}{2}}(y)} e^{n\hat{u}_k} \mathrm{d}\mu_{\hat{g}_k} \le \int_{\hat{B}_1(0)} e^{n\hat{u}_k} \mathrm{d}\mu_{\hat{g}_k} = \frac{1}{8}.$$

Then due to the upper bound (3.9), there exists $k_1 > 0$ such that, for any $k \ge k_1$,

$$\alpha_{\lambda_k} \lambda_k \le 2(n-1)!|\mathbb{S}^n|.$$

Then, for $k \ge k_1$, we have

$$\int_{\hat{B}_{1/2}(y)} \hat{Q}_k^+ e^{n\hat{u}_k} \mathrm{d}\mu_{g_k} \le \alpha_{\lambda_k} \lambda_k \int_{\hat{B}_{1/2}(y)} e^{n\hat{u}_k} \mathrm{d}\mu_{g_k} \le \frac{1}{4}(n-1)!|\mathbb{S}^n| \tag{4.3}$$

where $\hat{Q}_k = \alpha_{\lambda_k} \hat{f}_k$. Given $R > 0$, define a smooth cut-off function Ψ_R satisfying

$$\begin{cases} \Psi_R(z) = 1 \text{ for } |z| \le \frac{R}{2} \\ \Psi_R(z) = 0 \text{ for } |z| \ge R \end{cases}$$

We also set

$$a_k = \frac{1}{|\hat{B}_R|_{\hat{g}_k}} \int_{\hat{B}_R} \hat{u}_k \mathrm{d}\mu_{\hat{g}_k}$$

$$v_k = \Psi_R \hat{u}_k + (1 - \Psi_R) a_k = a_k + \Psi_R(\hat{u}_k - a_k)$$

$$\hat{v}_k = v_k - a_k.$$

Notice that $\hat{v}_k$ is identically zero outside $\hat{B}_R$. With help of Lemma 3.4, we have

$$\int_{\hat{B}_{2R}} \sum_{j=1}^{n-1} |\nabla^j \hat{u}_k|^p \mathrm{d}\mu_{\hat{g}_k} \le C_R, \quad p \in (1, \frac{n}{n-1}). \tag{4.4}$$

Applying the Poincaré inequality and the estimate (4.4), there holds

$$\begin{aligned}\int_{\hat{B}_R} |\hat{v}_k|^p \mathrm{d}\mu_{\hat{g}_k} &= \int_{\hat{B}_R} \Psi_R^p |\hat{u}_k - a_k|^p \mathrm{d}\mu_{\hat{g}_k} \leq \int_{\hat{B}_R} |\hat{u}_k - a_k|^p \mathrm{d}\mu_{\hat{g}_k} \\ &\leq \int_{B_R} |\nabla \hat{u}_k|^p \mathrm{d}\mu_{\hat{g}_k} \leq C_R, \quad p \in (1, \frac{n}{n-1}).\end{aligned}$$

Meanwhile, there holds

$$P_{\hat{g}_k} \hat{v}_k = \Psi_R P_{\hat{g}_k} \hat{u}_k + L_k(\hat{u}_k - a_k) = \Psi_R \hat{Q}_k e^{nu_k} + L_k(\hat{u}_k - a_k)$$

where $L_k(\hat{u}_k - a_k)$ are linear operators which contains derivatives of order $0 \leq j \leq n-1$ with uniformly bounded and smooth coefficients. Then

$$\int_{\hat{B}_{2R}} |L_k(\hat{u}_k - a_k)|^p \mathrm{d}\mu_{\hat{g}_k} \leq C_R, \quad p \in (1, \frac{n}{n-1}).$$

Hence using (4.3) and Remark 3.3, there exists $q > 1$ such that

$$\int_{\hat{B}_R} e^{nq\hat{v}_k} \mathrm{d}\mu_{\hat{g}_k} \leq C \tag{4.5}$$

due to (4.3). Actually, since $\hat{v}_k$ vanish identically outside $\hat{B}_R$, we can embed a fixed neighborhood of $(\hat{B}_{2R}, \hat{g}_k)$ into a compact manifold, a torus for example, such its metric converges to a flat one.

On the other hand,

$$a_k = \frac{1}{|\hat{B}_R|_{\hat{g}_k}} \int_{\hat{B}_R} \hat{u}_k \mu_{\hat{g}_k} \leq \frac{1}{n|\hat{B}_R|_{\hat{g}_k}} \int_{\hat{B}_R} e^{n\hat{u}_k} \mu_{\hat{g}_k} \leq C.$$

Since $v_k = \hat{u}_k$ in $\hat{B}_R$, there holds

$$\frac{1}{8} = \int_{\hat{B}_1(0)} e^{n\hat{u}_k} \mu_{\hat{g}_k} \leq e^{na_k} \int_{\hat{B}_R} e^{n\hat{v}_k} \mathrm{d}\mu_{\hat{g}_k} \leq C e^{na_k}.$$

Thus

$$|a_k| \leq C.$$

Combing with (4.5), one has

$$\int_{\hat{B}_R} e^{nq\hat{u}_k} \mathrm{d}\mu_{\hat{g}_k} \leq C$$

for some $q > 1$. Thus standard elliptic theory yields that $\hat{u}_k$ is bounded in $W^{4,q}(\hat{B}_{R/2})$. By the arbitrary choice of R, up to a subsequence, Sobolev embedding theorem shows that $(\hat{u}_k)$ converge strongly in $C^{\alpha}_{loc}(\mathbb{R}^n)$ for some $\alpha \in (0, 1)$ and also strongly in $H^{\frac{n}{2}}_{loc}(\mathbb{R}^n)$ to a function $\hat{u}_\infty \in C^{\alpha}_{loc}(\mathbb{R}^n) \cap H^{\frac{n}{2}}_{loc}(\mathbb{R}^n)$.

Firstly, by the Definition 1.1, we have

$$\begin{aligned}\alpha_{\lambda_k}(\lambda_k + \hat{f}_0) =&\alpha_{\lambda_k}\lambda_k(1 + \lambda_k^{-1} p_l(z_k + r_k z) + \lambda_k^{-1} O(|z_k + r_k z|^{l+1})) \\ =&\alpha_{\lambda_k}\lambda_k(1 + p_l(\lambda_k^{-\frac{1}{l}} z_k + \lambda_k^{-\frac{1}{l}} r_k z) + \lambda_k^{-1}|z_k + r_k z|^l O(|z_k + r_k z|))\end{aligned}$$

where $p_l(z)$ is a homogeneous function of degree l satisfying $p_l(z) \leq 0$. With help of Lemma 3.5 and the fact $z_k = \exp_P^{-1}(x_k)$, there holds

$$\lambda_k^{-1} r_k^l \leq C, \quad \lambda_k^{-1}|z_k|^l = \lambda_k^{-1} d_g(x_k, P)^l \leq C.$$

Hence, for $z \in \hat{B}_R(0)$, we have

$$\lambda_k^{-1}|z_k + r_k z|^l O(|z_k + r_k z|) \to 0, \quad \text{as } \lambda_k \to 0.$$

Due to Lemma 3.2 and (3.9), one may suppose that, up to a subsequence,

$$\lambda_k \alpha_{\lambda_k} \to \mu$$

where μ satisfies

$$\frac{1}{2}(n-1)!|\mathbb{S}^n| \leq \mu \leq (n-1)!|\mathbb{S}^n|. \tag{4.6}$$

With help of Lemma 3.5, up to a subsequence, there holds

$$\lim_{k\to\infty} \frac{r_k^l}{\lambda_k} \to r_0$$

where r_0 is a constant and satisfies $r_0 \geq 0$. In addition, up to a subsequence, we have

$$\lim_{k\to\infty} \lambda_k^{-\frac{1}{l}} z_k \to z_0 \in \mathbb{R}^n.$$

There are two possible cases including $r_0 > 0$ and $r_0 = 0$. We will discuss them one by one.

Case 1: $r_0 > 0$.

Then one has

$$\lim_{k\to\infty} \alpha_{\lambda_k}(\lambda_k + \hat{f}_0) = \mu(1 + p_l(z_0 + r_0 z))$$

uniformly on $\hat{B}_R(0)$. Then $\hat{u}_\infty \in H^{n/2}_{loc}(\mathbb{R}^n)$ weakly solve the equation

$$(-\Delta)^{\frac{n}{2}} \hat{u}_\infty = \mu(1 + p_l(z_0 + r_0 z)) e^{n\hat{u}_\infty} \tag{4.7}$$

satisfying

$$\int_{\mathbb{R}^n} e^{n\hat{u}_\infty} \mathrm{d}z \leq 1, \quad \int_{\mathbb{R}^n} |\mu(1 + p_l(z_0 + r_0 z))| e^{n\hat{u}_\infty} \mathrm{d}z < +\infty.$$

Meanwhile, with help of Lemma 3.4, for any $R > 0$, there holds

$$\int_{B_R(0)} |\Delta \hat{u}_\infty| \mathrm{d}z \leq C R^{n-2}$$

which concludes that

$$\frac{1}{|B_R(0)|} \int_{B_R(0)} |\Delta \hat{u}_\infty| \mathrm{d}z \to 0, \quad \text{as} \quad R \to \infty.$$

By a simple translation $w(z) = \hat{u}_\infty(z - \frac{1}{r_0} z_0) + \frac{1}{n} \log \mu$, it is not hard to check

$$(-\Delta)^{\frac{n}{2}} w(z) = (1 + p_l(r_0 z)) e^{n w(z)}$$

satisfying

$$\int_{\mathbb{R}^n} e^{nw} \mathrm{d}z \leq \mu, \quad \int_{\mathbb{R}^n} |(1 + p_l(r_0 z))| e^{nw} \mathrm{d}z < +\infty \tag{4.8}$$

and

$$\frac{1}{|B_R(0)|} \int_{B_R(0)} |\Delta w| \mathrm{d}z \to 0, \quad \text{as} \quad R \to \infty.$$

With help of Theorem 2.2 in [19], we show that $w(z)$ is a normal solution. Then applying higher order Bol's inequality Lemma 4.1, there holds

$$\int_{\mathbb{R}^n} e^{nw} \mathrm{d}z > (n-1)! |\mathbb{S}^n|$$

which contradicts to (4.6) and (4.8). Thus we rule out this 'slow bubble' case which means that $r_0 > 0$ is impossible.

Case 2: $r_0 = 0$.

On $\hat{B}_R(0)$, up to a subsequence, there holds

$$p_l(\lambda_k^{-\frac{1}{l}} z_k + \lambda_k^{-\frac{1}{l}} r_k z) \to \mu_0 \leq 0. \tag{4.9}$$

Thus $\hat{u}_\infty$ weakly solves the following equation

$$(-\Delta)^{\frac{n}{2}} \hat{u}_\infty = \mu(1 + \mu_0) e^{n\hat{u}_\infty} \quad \text{on} \quad \mathbb{R}^n$$

with the volume

$$\int_{\mathbb{R}^n} e^{n\hat{u}_\infty} \mathrm{d}z \leq 1. \tag{4.10}$$

Same as before, Lemma 3.4 shows that

$$\int_{B_R(0)} |\Delta \hat{u}_\infty| \mathrm{d}z = o(R^n).$$

Theorem 2.2 in [19] shows that $\hat{u}_\infty$ is a normal solution. With help of Lemma 2.18 in [19], one has $\mu_0 > -1$. Due to the classification theorem in [22, 31], and [24], one has

$$\mu(\mu_0 + 1) \int_{\mathbb{R}^n} e^{n\hat{u}_\infty} \mathrm{d}z = (n-1)!|\mathbb{S}^n|.$$

With help of (4.6), (4.10) and (4.9), we have

$$\mu_0 = 0, \quad \mu = (n-1)!|\mathbb{S}^n|, \quad \int_{\mathbb{R}^n} e^{n\hat{u}_\infty} \mathrm{d}z = 1.$$

In addition,

$$\hat{u}_\infty - \frac{1}{n} \log \frac{1}{|\mathbb{S}^n|} = \log \frac{2s}{s^2 + |z - z_0|^2}$$

for some $s > 0$ and $z_0 \in \mathbb{R}^n$. Moreover, there holds

$$\lim_{k \to \infty} \lambda_k \alpha_{\lambda_k} = (n-1)!|\mathbb{S}^n|.$$

Due to (2.12) and Lemma 2.4, up to a subsequence, there holds

$$\frac{n}{2} \lim_{k \to \infty} \frac{\beta(\lambda_k)}{\log 1/\lambda_k} = (n-1)!|\mathbb{S}^n|.$$

Using the unit volume constraint, there must be one blow up point denoted as p_0.

For any $x \in M \backslash \{p_0\}$, there exist $r > 0$ such that $B_r(x) \subset\subset M \backslash \{p_0\}$. Applying Lemma 3.1, there holds

$$u_k(x) \leq C + \bar{u}_k(x).$$

Combing with Lemma 3.2, we obtain that

$$u_k(x) \to -\infty, \quad \text{as } k \to \infty.$$

Finally, we finish our proof. □

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