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A REMARK ON THE CASE-GURSKY-VÉTOIS IDENTITY AND ITS APPLICATIONS

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ABSTRACT. Based on the works of Gursky [Comm. Math. Phys. 189 (1997), pp. 655–665], Vétois [Potential Anal. 61 (2024), pp. 485–500] and Case [J. Reine Angew. Math. 815 (2024), pp. 23–40], we make use of an Obata-type formula established in these works to obtain some Liouville-type theorems on conformally Einstein manifolds related to fourth-order Q-curvature. In particular, we solve a conjecture of Hang-Yang [Int. Math. Res. Not. IMRN 11 (2020), pp. 3295–3317] via an Obata-type argument and obtain optimal perturbation.

1. Introduction

Given a Riemannian manifold (M^n, g) with $n \geq 3$, Branson's Q-curvature [2] is defined as:

$$Q_g := -\frac{1}{2(n-1)}\Delta_g R_g - \frac{2}{(n-2)^2}|E_g|_g^2 + \frac{n^2-4}{8n(n-1)^2}R_g^2, \tag{1.1}$$

where R_g is the scalar curvature and E_g is the trace-free Ricci tensor, defined by $E_g := Ric_g - \frac{R_g}{n}g$. The Schouten tensor is given by $A_g := \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}g\right)$ and $\sigma_k(A_g)$ denotes the k-th symmetric function of the eigenvalues of A_g. Using this, we can rewrite Q-curvature as follows

$$Q_g = -\Delta_g \sigma_1(A_g) + 4\sigma_2(A_g) + \frac{n-4}{2}\sigma_1^2(A_g). \tag{1.2}$$

The remarkable Paneitz operator [20] is expressed as

$$P_g\varphi = \Delta_g^2\varphi + div_g\left\{(4A_g - (n-2)\sigma_1(A_g)g)(\nabla\varphi, \cdot)\right\} + \frac{n-4}{2}Q_g\varphi. \tag{1.3}$$

For $n \geq 3$ and $n \neq 4$, the Q-curvature $Q_{\tilde{g}}$ of the conformal metric $\tilde{g} = \varphi^{\frac{4}{n-4}}g$ satisfies the following conformally invariant equation

$$P_g\varphi = \frac{n-4}{2}Q_{\tilde{g}}\varphi^{\frac{n+4}{n-4}}. \tag{1.4}$$

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For $n = 4$, the fourth-order Q-curvature $Q_{\tilde{g}}$ of the conformal metric $\tilde{g} = e^{2\varphi}g$ satisfies

$$P_g\varphi + Q_g = Q_{\tilde{g}}e^{4\varphi}. \tag{1.5}$$

Similar to the Yamabe problem, it is interesting to identify a conformal metric such that the Q-curvature is constant. For $n = 4$, significant advancements were made by Chang and Yang [6] and Djadli and Malchiodi [7]. In dimensions $n \geq 3$ and $n \neq 4$, Gursky and Malchiodi [14] and Hang and Yang [16], [17] established existence results under suitable conditions related to positivity of Q-curvature and scalar curvature or Yamabe invariant. More related discussions can be found in [13], [21], and the references therein.

The well known Obata's theorem (see [19]) states that for an Einstein manifold (M^n, g_0) with $n \geq 3$, if the scalar curvature of the conformal metric $g = u^2 g_0$ is constant, g must also be Einstein. Furthermore, if (M, g_0) is not conformally equivalent to round sphere, u must be constant.

When g_0 is Einstein, the Obata trick is that the trace-free Ricci tensor of the conformal metric $g = u^2 g_0$ can be written as

$$u(E_g)_{ij} = -(n-2)\left(\nabla^2_{ij}u - \frac{\Delta_g u}{n}g_{ij}\right). \tag{1.6}$$

Here the covariant derivatives are with respect to the metric g. Multiplying $(E_g)_{ij}$ with (1.6), integrating over (M^n, g), using integration by parts, and then using the contracted second Bianchi identity,

$$\nabla_i(E_g)_{ij} = \frac{n-2}{2n}\nabla_j R_g, \tag{1.7}$$

yields the famous Obata identity

$$\int_M u|E_g|^2 \mathrm{d}v_g = \frac{(n-2)^2}{2n}\int_M \langle \nabla R_g, \nabla u\rangle_g \mathrm{d}v_g. \tag{1.8}$$

Using (1.8), it is easy to see that if R_g is constant, g must be Einstein.

Given an Einstein manifold (M^n, g_0) with scalar curvature R_0, the Paneitz operator (1.3) can be succinctly expressed as follows (see [10], [8] for more details)

$$P_{g_0}\varphi = \Delta^2_{g_0}\varphi - \frac{n^2-2n-4}{2n(n-1)}R_0\Delta_{g_0}\varphi + \frac{n-4}{2}Q_0\varphi, \tag{1.9}$$

where the Q-curvature Q_0 is given by

$$Q_0 = \frac{n^2-4}{8n(n-1)^2}R_0^2. \tag{1.10}$$

It is natural to ask whether a similar Obata-type theorem holds for Q-curvature? Firstly, Vétois [22] made use of Bochner–Lichnerowicz–Weitzenbock formula and an important Lemma 2.5 established by Gursky and Malchiodi [14] concerning the positivity of scalar curvature to deal with the equations (1.4) and (1.5). Then he established the following Obata-type theorem for Q-curvature.

Theorem 1.1 (Vétois' theorem in [22])**.** *Suppose that (M^n, g_0) where $n \geq 3$ is a compact Einstein manifold with non-negative scalar curvature R_0. If the fourth-order Q-curvature of conformal metric $g = u^2 g_0$ is constant, then g is Einstein. Furthermore, if (M^n, g_0) is not conformally equivalent to a round sphere, u must be constant.*

Very recently, Case [4] considered a more general Obata-Vétois type theorem. Given a compact manifold (M^n, g) and a constant a, Case considered the following mixed curvature

$$I_a(g) := Q_g + a\sigma_2(A_g), \tag{1.11}$$

where Q_g and A_g are the Q-curvature and Schouten tensor respectively given before. This mixed curvature $I_a(g)$ arises naturally in the study of conformally variational Riemannian invariants. For more details, we refer to [5] and references therein.

Theorem 1.2 (Case's theorem in [4])**.** *Given a compact Einstein manifold (M^n, g_0) where $n \geq 3$ with non-negative scalar curvature R_0. Suppose that the constant a satisfies*

$$\frac{(n-2)(n-4) - n\sqrt{n^2+4n}}{2(n-1)} \leq a \leq \frac{(n-2)(n-4) + n\sqrt{n^2+4n}}{2(n-1)}. \tag{1.12}$$

Consider a conformal metric given by $g = u^2 g_0$. If $I_a(g)$ is a constant and the scalar curvature $R_g \geq 0$, then g is Einstein.

Remark 1.3. In the original version of Case's theorem, the constant a is confined to a smaller interval. By applying the sharp inequality from Lemma 2.3, we can extend the range of the constant a slightly. Nevertheless, identifying the optimal range remains an intriguing open question.

To prove Theorem 1.2, Case established a remarkable identity (see Lemma 3.2 in [4]) as follows

$$\begin{aligned} 0 =& \frac{1}{2}\int_M u|\nabla R_g|_g^2 \mathrm{d}v_g + \frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g, \nabla u)\mathrm{d}v_g \\ &+ \frac{n(n-1)^2(a+4)}{2(n-2)^2}\int_M |E_g|_g^2 u^{-1}|\nabla u|_g^2 \mathrm{d}v_g \\ &+ \frac{(n-1)(a+4)+2n^2-4n}{2(n-2)^2}\int_M u|E_g|_g^2 R_g \mathrm{d}v_g \\ &+ \frac{(n-1)(a+4)R_0}{2(n-2)^2}\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g \end{aligned} \tag{1.13}$$

under the assumption that $I_a(g)$ is constant. Such formula is also obtained by Gursky (see the equation (1.13) in [11]) on $\mathbb{S}^4$ and Vétois (see equation (1.4) in [22]) for $a = 0$.

As noted by Case [4], the condition $R_g \geq 0$ in Theorem 1.2 is somewhat unsatisfying. When $a = 0$, Gursky and Malchiodi's continuity method [14] guarantees that $R_g \geq 0$ holds automatically. However, for $a \neq 0$, the situation changes significantly. To address this, Case [4] first demonstrated that when $R_0 \geq 0, n \geq 4$, and $-4 \leq a \leq 0$, if $I_a(g)$ is a constant, then $I_a(g)$ must be non-negative (see Proposition 4.2 in [4]) by using the representation of Paneitz operator (1.9) on Einstein manifolds. He then adapted Gursky's observation (Lemma 2.1 in [12]) to show that if $I_a(g) \geq 0, R_0 \geq 0$, and $a \leq -\frac{2(n-2)}{n-1}$, then the scalar curvature of the conformal metric $R_g \geq 0$ (see Lemma 4.3 in [4]). Thus, under the assumptions that $R_0 \geq 0$, a meets the criteria (1.12), along with $a \leq -\frac{2(n-2)}{n-1}$ and $n \geq 4$, Case removed the assumption $R_g \geq 0$ in Theorem 1.2. For $n = 3$, Case additionally assumed that $I_a(g) \geq 0$.

In summary, the sign of $I_a(g)$ is crucial. In this paper, we utilize Obata's identity (1.8) to ascertain the sign of $I_a(g)$. Unlike Case's method, our approach allows us to address the situation for $n=3$.

Theorem 1.4. *Suppose that (M^n, g_0) where $n \geq 3$ is a compact Einstein manifold with constant scalar curvature R_0. Consider a conformal metric given by $g = u^2 g_0$. For any constant $a \in [-\frac{n^2-4}{n-1}, -\frac{2(n-2)}{n-1}]$, if $I_a(g)$ is constant, then it holds that*

$$I_a(g) \geq 0$$

with equality holds if and only if one of the following conditions is satisfied

(1) $a \in (-\frac{n^2-4}{n-1}, -\frac{2(n-2)}{n-1}]$, $R_0 = 0, u \equiv C$
(2) $a = -\frac{n^2-4}{n-1}$ *and g is Einstein.*

Remark 1.5. In the theorem above, we don't need to assume that $R_0 \geq 0$. A direct computation yields that $I_a(g_0) = \frac{(n-1)a+n^2-4}{8(n-1)^2 n} R_0^2$ which indicates that $-\frac{n^2-4}{n-1}$ is the sharp lower bound of a such that $I_a(g) \geq 0$.

Combining Theorem 1.4 with Lemma 4.3 in [4], we find that the assumption $R_g \geq 0$ in Theorem 1.2 holds automatically under a suitable range of a. In particular, we generalize Theorem 1.2 in [4] when $n=3$.

Corollary 1.6. *Suppose that (M^n, g_0) where $n \geq 3$ is an Einstein manifold with the scalar curvature $R_0 \geq 0$ and the constant a satisfies*

$$\max\{\frac{(n-2)(n-4) - n\sqrt{n^2+4n}}{2(n-1)}, -\frac{n^2-4}{n-1}\} \leq a \leq -\frac{2(n-2)}{n-1}. \tag{1.14}$$

Consider a conformal metric given by $g = u^2 g_0$. If $I_a(g)$ is a constant, then g is Einstein.

In [22], Vétois established a Liouville-type theorem by considering the following equations

$$P_{g_0}\varphi = \varphi^p, \quad n \neq 4,\ p \leq \frac{n+4}{n-4}, \tag{1.15}$$

$$P_{g_0}\varphi = e^{p\varphi}, \quad n = 4,\ p \leq 4, \tag{1.16}$$

on compact Einstein manifolds. He showed that the positive smooth solutions to (1.15) and smooth solutions to (1.16) must be constant if (M^n, g) is not conformally equivalent to the round sphere.

Building on the works of Gursky [11], Vétois [22], and Case [4], we find that we can reintegrate the term $I_a(g)$ into formula (1.13) without requiring that $I_a(g)$ be constant. This modified formula will be crucial in the proof of Liouville-type theorems. Overall, the Case-Gursky-Vétois identity is encapsulated in Theorem 2.1. Utilizing this identity allows us to streamline the proofs of the Liouville-type theorems established by Vétois (see Theorems 2.1 and 2.2 in [22]). Moreover, we can extend the Liouville-type results discussed in [1], [3], and [9]—which are related to second-order nonlinear equations—to fourth-order cases by examining perturbations of the linear term in the Paneitz operator.

Theorem 1.7. *Suppose that (M^n, g_0) where $n \geq 3$ is a compact Einstein manifold with positive scalar curvature R_0.*

(1) *When $n \neq 4$, consider the positive smooth solution φ to the equation*

$$P_{g_0}\varphi - \frac{n-4}{2}\varepsilon\varphi = \frac{n-4}{2}\varphi^{\frac{(n+4)p}{n-4}}, \tag{1.17}$$

where $0 \leq \varepsilon < Q_0$ and $p \leq 1$.

(2) *When $n = 4$, consider the smooth solution φ to the equation*

$$P_{g_0}\varphi + Q_0 - \varepsilon = e^{4p\varphi}, \tag{1.18}$$

where $0 \leq \varepsilon < Q_0$ and $p \leq 1$.

If (M^n, g_0) is conformally equivalent to round sphere, we further assume that $\varepsilon > 0$ or $p < 1$. Then φ must be constant.

Remark 1.8. By using the representation (1.9) and integrating the equation (1.17) or (1.18) over (M^n, g_0), it is easy to see that the condition $\varepsilon < Q_0$ is necessary for the existence of positive solutions. When $n = 3$, $p = 1$, $0 < \varepsilon < Q_0$, and (M^3, g_0) is the standard sphere, Theorem 1.7 recovers Conjecture 1.1 of Hang and Yang [15] through an Obata-type argument. Initially, Zhang [23] solved this conjecture by transforming the equation on the sphere into Euclidean space and applying the moving plane method. Subsequently, Hyder and Ngô [18] generalized this theorem to higher-order cases using a similar moving plane approach. However, it is important to note that their methods can only accommodate small values of ε due to their reliance on a compactness theorem. In contrast, our use of the Obata-type argument allows us to determine the optimal range for ε.

In fact, with the help of the strong maximum principle established by Gursky and Malchiodi (Theorem 2.2 in [14]) and Hang-Yang (Proposition 2.1 in [17]), we are able to establish a more general result as below.

Theorem 1.9. *Suppose that (M^n, g_0) where $n \geq 3$ is a compact Einstein manifold with positive scalar curvature R_0.*

(1) *When $n \neq 4$, consider the solution $\varphi \in C^4(M^n, g_0)$ to the following equation*

$$P_{g_0}\varphi = \frac{n-4}{2}f(\varphi),$$

where $f(t) \geq 0$ for all $t \in \mathbb{R}$ is a smooth function satisfying

$$\frac{n-4}{2}\partial_t\left(t^{-\frac{n+4}{n-4}}f(t)\right) \leq 0, \quad \forall\, t > 0. \tag{1.19}$$

(2) *When $n = 4$, consider the solution $\varphi \in C^4(M^4, g_0)$ to the following equation*

$$P_{g_0}\varphi + Q_0 = f(e^{\varphi}),$$

where $f(t) \geq 0$ for all $t > 0$ is a smooth function satisfying

$$\partial_t\left(t^{-4}f(t)\right) \leq 0, \quad \forall\, t > 0. \tag{1.20}$$

If (M^n, g_0) is conformally equivalent to the round sphere, we additionally assume that the inequality in (1.19) and (1.20) is strict. Then φ must be constant.

This paper is organized as follows. In Section 2, we prove the Case-Gursky-Vétois identity by following the argument of Gursky [11] and Case [4]. With the help of such identity, we establish two Case-Gursky-Vétois rigidity inequalities and then repeat the proofs of Theorem 1.1 and Theorem 1.2. Then, in Section 3, we use Obata identity to determine the sign of constant $I_a(g)$ and prove Theorem 1.4. Finally, in Section 4, with the help of rigidity inequalities, we provide the proofs of Theorem 1.7 and 1.9.

2. Case-Gursky-Vétois identity on conformally Einstein manifolds

First, we introduce some notations for future reference. Notice that

$$\sigma_2(A_g) = -\frac{|E_g|_g^2}{2(n-2)^2} + \frac{R_g^2}{8(n-1)n}. \tag{2.1}$$

With the help of (1.1) and (1.11), one has

$$2(n-1)I_a(g) = -\Delta_g R_g - \alpha_1 |E_g|_g^2 + \alpha_2 R_g^2 \tag{2.2}$$

where the constants α_1 and α_2 satisfy

$$\alpha_1 = \frac{(n-1)(4+a)}{(n-2)^2}, \quad \alpha_2 = \frac{(n-1)a + n^2 - 4}{4(n-1)n}. \tag{2.3}$$

Now, we are going to prove Case-Gursky-Vétois identity (2.4). During the proof of this formula, we basically follow Case [4], using the idea of Gursky [11]. The key observation during Gursky's proof (1997, CMP, Page 660) is that integrating $u\langle E_g, \nabla^2 R_g\rangle_g$ over (M^n, g), then inserting the representations of $\Delta_g u$ and $\Delta_g R_g$. We should point out that this identity can also be obtained by inserting the tensor T defined in Page 4 of [4] into the equation (1.6) without assuming $I_a(g)$ is a constant.

Theorem 2.1 (Case-Gursky-Vétois identity)**.** *Suppose that (M^n, g_0) where $n \geq 3$ is an Einstein manifold with constant scalar curvature R_0. Consider a smooth conformal metric given by $g = u^2 g_0$. Then there holds*

$$\begin{aligned} &2(n-1)^2 \int_M \langle \nabla I_a(g), \nabla u\rangle_g \mathrm{d}v_g \\ =&\frac{1}{2}\int_M u|\nabla R_g|_g^2 \mathrm{d}v_g + \frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g, \nabla u)\mathrm{d}v_g \\ &+ \frac{n(n-1)^2(a+4)}{2(n-2)^2}\int_M |E_g|_g^2 u^{-1}|\nabla u|_g^2 \mathrm{d}v_g \\ &+ \frac{(n-1)(a+4)+2n^2-4n}{2(n-2)^2}\int_M u|E_g|_g^2 R_g \mathrm{d}v_g \\ &+ \frac{(n-1)(a+4)R_0}{2(n-2)^2}\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g. \end{aligned} \tag{2.4}$$

Proof. Since g_0 is an Einstein metric, the equation (1.6) holds and we have

$$\Delta_g u = \frac{n}{2}u^{-1}|\nabla_g u|_g^2 - \frac{R_g u}{2(n-1)} + \frac{u^{-1}R_0}{2(n-1)}. \tag{2.5}$$

Multiplying $\nabla_{ij}^2 R_g$ on both sides of the equations (1.6) and integrating it over (M^n, g), then there holds

$$\int_M u\langle E_g, \nabla^2 R_g\rangle_g \mathrm{d}v_g = -(n-2)\int_M \langle \nabla^2 u, \nabla^2 R_g\rangle_g \mathrm{d}v_g + \frac{n-2}{n}\int_M \Delta_g u \cdot \Delta_g R_g \mathrm{d}v_g. \tag{2.6}$$

We are going to deal with these three terms one by one.

Firstly, with the help of second Bianchi identity (1.7) and integration by parts, one has

$$\int_M u\langle E_g, \nabla^2 R_g\rangle_g \mathrm{d}v_g = -\frac{n-2}{2n}\int_M u|\nabla R_g|_g^2 \mathrm{d}v_g - \int_M E_g(\nabla u, \nabla R_g)\mathrm{d}v_g. \tag{2.7}$$

Secondly, integrating by parts and using Ricci identity, one has

$$\begin{aligned}
&\int_M \langle \nabla^2 u, \nabla^2 R_g\rangle_g \mathrm{d}v_g\\
=&-\int_M \langle \nabla R_g, \nabla \Delta_g u\rangle_g \mathrm{d}v_g - \int_M Ric_g(\nabla u, \nabla R_g)\mathrm{d}v_g\\
=&\int_M \Delta_g R_g\cdot \Delta_g u \mathrm{d}v_g - \int_M E_g(\nabla u, \nabla R_g)\mathrm{d}v_g - \frac{1}{n}\int_M R_g\langle \nabla R_g, \nabla u\rangle_g \mathrm{d}v_g.
\end{aligned}$$

Combining these formulas, we rewrite (2.6) as follows

$$\begin{aligned}
0=&-\frac{1}{2n}\int_M u|\nabla R_g|^2 \mathrm{d}v_g - \frac{n-1}{n-2}\int_M E_g(\nabla u, \nabla R_g)\mathrm{d}v_g\\
&+\frac{n-1}{n}\int_M \Delta_g R_g\cdot\Delta_g u\mathrm{d}v_g - \frac{1}{n}\int_M R_g\langle\nabla R_g, \nabla u\rangle_g \mathrm{d}v_g.
\end{aligned} \tag{2.8}$$

With the help of the identities (2.2) and (2.5), one has

$$\begin{aligned}
&\int_M \Delta_g R_g\cdot \Delta_g u\mathrm{d}v_g\\
=&\int_M \left(-2(n-1)I_a(g) - \alpha_1|E_g|_g^2 + \alpha_2 R_g^2\right)\cdot \Delta_g u \mathrm{d}v_g\\
=&2(n-1)\int_M \langle \nabla I_a, \nabla u\rangle_g \mathrm{d}v_g - \alpha_1\int_M |E_g|_g^2\left(\frac{n}{2}u^{-1}|\nabla u|_g^2 - \frac{R_g u}{2(n-1)} + \frac{R_0 u^{-1}}{2(n-1)}\right)\mathrm{d}v_g\\
&-2\alpha_2\int_M R_g\langle \nabla R_g, \nabla u\rangle_g \mathrm{d}v_g.
\end{aligned}$$

Inserting the above formula into (2.8), we obtain that

$$\begin{aligned}
&2(n-1)^2\int_M \langle \nabla I_a, \nabla u\rangle_g \mathrm{d}v_g\\
=&\frac{1}{2}\int_M u|\nabla R_g|_g^2 \mathrm{d}v_g + \frac{n(n-1)}{n-2}\int_M E_g(\nabla R_g, \nabla u)\mathrm{d}v_g\\
&+\frac{n(n-1)\alpha_1}{2}\int_M |E_g|_g^2 u^{-1}|\nabla u|_g^2 \mathrm{d}v_g - \frac{\alpha_1}{2}\int_M |E_g|_g^2 R_g u \mathrm{d}v_g\\
&+\frac{\alpha_1 R_0}{2}\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g + (2(n-1)\alpha_2 + 1)\int_M R_g\langle \nabla R_g, \nabla u\rangle_g \mathrm{d}v_g.
\end{aligned} \tag{2.9}$$

Multiplying $R_g E_{ij}$ on both sides of (1.6) and integrating by parts, there holds

$$\begin{aligned}
\int_M uR_g|E_g|_g^2 \mathrm{d}v_g =&-(n-2)\int_M R_g\langle E_g, \nabla^2 u\rangle_g \mathrm{d}v_g\\
=&\frac{(n-2)^2}{2n}\int_M R_g\langle \nabla R_g, \nabla u\rangle_g \mathrm{d}v_g + (n-2)\int_M E_g(\nabla R_g, \nabla u)\mathrm{d}v_g
\end{aligned}$$

which is equivalent to

(2.10)
$$\int_M R_g\langle\nabla R_g,\nabla u\rangle_g \mathrm{d}v_g=\frac{2n}{(n-2)^2}\int_M uR_g|E_g|_g^2\mathrm{d}v_g-\frac{2n}{n-2}\int_M E_g(\nabla R_g,\nabla u)\mathrm{d}v_g.$$

By inserting the identity (2.10) and the notations (2.3) into (2.9), we establish our desired identity (2.4). This completes the proof. □

With the help of above identity, we are able to establish the following Case-Gursky-Vétois inequalities.

Corollary 2.2. *Suppose that (M^n,g_0) where $n\geq 3$ is a compact Einstein manifold with constant scalar curvature R_0. Consider a smooth conformal metric given by $g=u^2g_0$. Then there holds*
$$(n-2)^2(n-1)\int_M\langle\nabla Q_g,\nabla u\rangle_g\mathrm{d}v_g\geq\frac{n^2-2}{2(n-1)}\int_M u|E_g|_g^2R_g\mathrm{d}v_g+R_0\int_M|E_g|_g^2u^{-1}\mathrm{d}v_g$$
with the equality holds if and only if g is Einstein.

Proof. Choosing $a=0$ in Theorem 2.1, one has
$$\begin{aligned}2(n-1)^2&\int_M\langle\nabla Q_g,\nabla u\rangle_g\mathrm{d}v_g\\ =&\frac{1}{2}\int_M u|\nabla R_g|_g^2\mathrm{d}v_g+\frac{4-3n}{n-2}\int_M E_g(\nabla R_g,\nabla u)\mathrm{d}v_g\\ &+\frac{2n(n-1)^2}{(n-2)^2}\int_M|E_g|_g^2u^{-1}|\nabla u|_g^2\mathrm{d}v_g\\ &+\frac{n^2-2}{(n-2)^2}\int_M u|E_g|_g^2R_g\mathrm{d}v_g+\frac{2(n-1)R_0}{(n-2)^2}\int_M|E_g|_g^2u^{-1}\mathrm{d}v_g.\end{aligned}\tag{2.11}$$

With help of Cauchy inequality and Young's inequality, one has
$$\frac{4-3n}{n-2}E_g(\nabla R_g,\nabla u)\geq-\frac{2n(n-1)^2}{(n-2)^2}|E_g|_g^2u^{-1}|\nabla u|_g^2-\frac{(3n-4)^2}{8n(n-1)^2}u|\nabla R_g|_g^2.$$

Inserting it into the above identity (2.11), one has
$$\begin{aligned}2(n-1)^2&\int_M\langle\nabla Q_g,\nabla u\rangle_g\mathrm{d}v_g\\ \geq&\frac{4(n-4)(n-1)^2+7n^2-8n}{8n(n-1)^2}\int_M u|\nabla R_g|_g^2\mathrm{d}v_g\\ &+\frac{n^2-2}{(n-2)^2}\int_M u|E_g|_g^2R_g\mathrm{d}v_g+\frac{2(n-1)R_0}{(n-2)^2}\int_M|E_g|_g^2u^{-1}\mathrm{d}v_g.\end{aligned}\tag{2.12}$$

Immediately, there holds

(2.13)
$$(n-2)^2(n-1)\int_M\langle\nabla Q_g,\nabla u\rangle_g\mathrm{d}v_g\geq\frac{n^2-2}{2(n-1)}\int_M u|E_g|_g^2R_g\mathrm{d}v_g+R_0\int_M|E_g|_g^2u^{-1}\mathrm{d}v_g.$$

On one hand, if the equality holds in (2.13), with the help of the inequality (2.12), one must have $|\nabla R_g|_g=0$ which means that the scalar curvature R_g is a constant. Then Obata's theorem shows that g is Einstein. On the other hand, if g is Einstein, it is not hard to check that the equality in (2.13) is achieved by using (1.10). Thus we finish our proof. □

To slightly extend Case's Theorem 1.1 in [4], we need the following well known lemma. For the reader's convenience, we sketch the proof here.

Lemma 2.3. *Let A be a $n\times n$ symmetric matrix satisfying $trace(A)=0$. Let x and y be two $n\times 1$ vectors. There holds*

$$|x^TAy|\le\sqrt{\frac{n-1}{n}}|A|\cdot|x|\cdot|y|.$$

Proof. With the help of orthogonality, one may assume that A is a diagonal matrix and let λ_i be the diagonal elements where $1\le i\le n$ and $|\lambda_i|\le|\lambda_1|$ for all $1\le i\le n$. Notice that

$$|A|^2=\sum_{i=1}^n\lambda_i^2\ge\lambda_1^2+\frac{1}{n-1}\left(\sum_{i=2}^n\lambda_i\right)^2=\frac{n}{n-1}\lambda_1^2. \tag{2.14}$$

With the help of (2.14) and Cauchy inequality, one has

$$|x^TAy|=|\sum_i\lambda_ix_iy_i|\le|\lambda_1|\cdot|x|\cdot|y|\le\sqrt{\frac{n-1}{n}}|A|\cdot|x|\cdot|y|.$$

□

Corollary 2.4. *Suppose that (M^n,g_0) where $n\ge3$ is a compact Einstein manifold with constant scalar curvature R_0. Consider a conformal metric given by $g=u^2g_0$. If the constant a satisfies (1.12), then there holds*

$$\int_M\langle\nabla I_a(g),\nabla u\rangle_g\mathrm{d}v_g\ge a_1\int_M u|E_g|^2R_g\mathrm{d}v_g+a_2R_0\int_M|E_g|^2u^{-1}\mathrm{d}v_g.$$

where the constants $a_1=\frac{(n-1)(a+4)+2n^2-4n}{4(n-2)^2(n-1)^2}$ and $a_2=\frac{a+4}{4(n-2)^2(n-1)}$.

Proof. With the help of Lemma 2.3 and Cauchy-Schwarz inequality as well as Young's inequality, there holds

$$\begin{aligned}&\frac{n-(n-1)(a+4)}{n-2}\int_M E_g(\nabla R_g,\nabla u)\mathrm{d}v_g\\ \ge&-|\frac{n-(n-1)(a+4)}{n-2}|\sqrt{\frac{n-1}{n}}\int_M|E_g|_g\cdot|\nabla R_g|_g\cdot|\nabla u|_g\mathrm{d}v_g\\ \ge&-\frac12\int_M u|\nabla R_g|_g^2\mathrm{d}v_g-\frac{\left(n-(n-1)(a+4)\right)^2(n-1)}{2(n-2)^2n}\int_M|E_g|_g^2u^{-1}|\nabla u|_g^2.\end{aligned}$$

Using the condition (1.12) and Theorem 2.1, we obtain our desired result. □

For the reader's convenience, we restate the lemma by Gursky and Malchiodi [14] as follows.

Lemma 2.5 (See Theorem 2.2 in [14], Proposition 2.1 in [17], Theorem 2.3 in [22])**.** *Given a compact manifold (M^n,g) where $n\ge3$ with positive scalar curvature and non-negative Q-curvature. Consider the conformal metric $\tilde g=u^2g$. If the Q-curvature $Q_{\tilde g}\ge0$, then the scalar curvature $R_{\tilde g}$ is positive.*

Now, we repeat the proof of Vétois' Theorem 1.1 and Case's Theorem 1.2 by using Case-Gursky-Vétois inequalities.

Proof of Theorem 1.1: When the scalar curvature $R_0 = 0$, then Q-curvature of g_0 also vanishes by (1.10). For $n \geq 3$ and $n \neq 4$, if the fourth-order Q-curvature of conformal metric of $g = \varphi^{\frac{4}{n-4}} g_0$ is constant, the equation (1.9) yields that

$$\Delta_{g_0}^2 \varphi = C\varphi^{\frac{n+4}{n-4}}, \tag{2.15}$$

where C is a constant. By integrating (2.15) over (M, g_0), it is easy to see that the constant C must be zero. Immediately, $\Delta_{g_0}\varphi$ must be a constant and such constant must be zero by integrating it over (M, g_0) again. Then φ must be a constant. For $n = 4$, consider $g = e^{2\varphi} g_0$ and the same argument yields that φ must be constant.

When $R_0 > 0$, with the help of (1.1), one has

$$Q_{g_0} = \frac{n^2-4}{8n(n-1)^2} R_0^2 > 0.$$

By integrating the equations (1.4) and (1.5) over (M^n, g_0), it is easy to see that the constant $Q_g > 0$. Applying Lemma 2.5, the scalar curvature R_g must be positive. Making use of Corollary 2.2 and the fact Q_g is a constant, one has

$$0 \geq \frac{n^2-2}{2(n-1)} \int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0 \int_M |E_g|_g^2 u^{-1} \mathrm{d}v_g.$$

Since u, R_g, R_0 are all positive, one must have $E_g \equiv 0$ which means that g is Einstein. Furthermore, if (M^n, g_0) is not conformally equivalent to round sphere, Obata's theorem yields that u must be constant.

Proof of Theorem 1.2: If $R_0 = 0$, one has

$$-\frac{4(n-1)}{n-2}\Delta_0 u^{\frac{n-2}{2}} = R_g u^{\frac{n+2}{2}} \geq 0.$$

Integrating it over (M^n, g_0), the left side vanishes and then we must have $R_g \equiv 0$. Immediately, u must be a constant by using Obata's theorem. Otherwise, if $R_0 > 0$, $R_g \geq 0$ and I_a is a constant, Corollary 2.4 yields that $E_g \equiv 0$ which means that g is Einstein.

3. Sign of $I_a(g)$ curvature

With the help of the Obata identity (1.8), we are going to determine the sign of $I_a(g)$.

Proof of Theorem 1.4: With the help of the representations (1.2), (2.1) and integration by parts, there holds

$$\begin{aligned}
&\int_M I_a(g) u \mathrm{d}v_g \\
=& \int_M (-\Delta_g \sigma_1(A_g)) u \mathrm{d}v_g + (4+a)\int_M \sigma_2(A_g) u \mathrm{d}v_g + \frac{n-4}{2}\int_M \sigma_1(A_g)^2 u \mathrm{d}v_g \\
=& \int_M \langle \nabla \sigma_1(A_g), \nabla u\rangle_g \mathrm{d}v_g - \frac{4+a}{2(n-2)^2}\int_M |E_g|_g^2 u \mathrm{d}v_g \\
&+ \frac{n^2-4+a(n-1)}{8(n-1)^2 n}\int_M R_g^2 u \mathrm{d}v_g.
\end{aligned}$$

Inserting Obata's identity (1.8) into the above formula, we obtain that

$$\int_M I_a(g)u\mathrm{d}v_g = \frac{-2(n-2)-a(n-1)}{2(n-1)(n-2)^2}\int_M |E_g|_g^2 u\mathrm{d}v_g + \frac{n^2-4+a(n-1)}{8(n-1)^2 n}\int_M R_g^2 u\mathrm{d}v_g. \tag{3.1}$$

If $a \in [-\frac{n^2-4}{n-1}, -\frac{2(n-1)}{n-2}]$, the above identity (3.1) yields that

$$\int_M I_a(g)u\mathrm{d}v_g \geq 0.$$

Immediately, if $I_a(g)$ is a constant, one has

$$I_a(g) \geq 0.$$

On one hand, if $I_a(g) = 0$ and $a \in (-\frac{n^2-4}{n-1}, -\frac{2(n-2)}{n-1}]$, using the identity (3.1), there holds $R_g \equiv 0$ which deduces that R_0 must also be zero. Then, Obata theorem yields that u must be a constant. On the other hand, if $I_a(g) = 0$ and $a = -\frac{n^2-4}{n-1}$, the identity (3.1) yields that $E_g \equiv 0$ which means that g is Einstein. Thus we finish our proof.

4. Applications on Liouville-type theorems

In this section, we use Corollary 2.2 to establish a series of Liouville-type theorems related to Paneitz operator on compact Einstein manifolds.

Proof of Theorem 1.7: For $n \neq 4$, consider the conformal metric $g = \varphi^{\frac{4}{n-4}} g_0$ and then the fourth-order Q-curvature satisfies

$$Q_g = \epsilon\varphi^{-\frac{8}{n-4}} + \varphi^{(p-1)\frac{n+4}{n-4}} > 0. \tag{4.1}$$

Due to the scalar curvature $R_0 > 0$, (1.10) and (4.1), Lemma 2.5 yields that

$$R_g > 0.$$

Setting $u = \varphi^{\frac{2}{n-4}}$ in Corollary 2.2 and using (4.1) again, a direct computation yields that

$$\begin{aligned}\langle \nabla Q_g, \nabla u\rangle_g =& -\frac{16\varepsilon}{(n-4)^2}\varphi^{\frac{2-2n}{n-4}}|\nabla\varphi|_g^2 - \frac{2(n+4)(1-p)}{(n-4)^2}\varphi^{\frac{p(n+4)-3(n-2)}{n-4}}|\nabla\varphi|_g^2\\ =& C(\varphi,\varepsilon,p,n)|\nabla\varphi|_g^2,\end{aligned}$$

where

$$C(\varphi,\varepsilon,p,n) := -\frac{16\varepsilon}{(n-4)^2}\varphi^{\frac{2-2n}{n-4}} - \frac{2(n+4)(1-p)}{(n-4)^2}\varphi^{\frac{p(n+4)-3(n-2)}{n-4}}.$$

If $\epsilon = 0$ and $p = 1$, based on our assumption, (M^n, g_0) is not conformally equivalent to a round sphere. In this situation, one has

$$C(\varphi,\varepsilon,p,n) = 0.$$

Then Corollary 2.2 yields that

$$0 \geq \int_M u|E_g|_g^2 R_g \mathrm{d}v_g + R_0\int_M |E_g|_g^2 u^{-1}\mathrm{d}v_g.$$

Noticing that $R_g > 0$ and $R_0 > 0$, one must have $E_g \equiv 0$ which yields that g has constant scalar curvature. Immediately, Obata theorem deduces that φ must be constant.

Otherwise, one must have $C(\varphi,\varepsilon,p,n)<0$. Using Corollary 2.2 and the facts the scalar curvatures R_g and R_0 are both positive, we have

$$\int_M C(\varphi,\varepsilon,p,n)|\nabla\varphi|_g^2 \mathrm{d}v_g \geq 0. \tag{4.2}$$

Immediately, one has $|\nabla\varphi|_g=0$ which means that φ must be constant.

For $n=4$, consider the conformal metric $g=e^{2\varphi}g_0$ and fourth-order Q-curvature satisfies $Q_g=\varepsilon e^{-4\varphi}+e^{(4(p-1)\varphi}>0$. Using the same argument as before, φ must be constant. Thus, we finish our proof.

Proof of Theorem 1.9: For $n\geq 5$, our assumption $f(t)\geq 0$ yields that

$$P_{g_0}\varphi\geq 0.$$

Then the strong maximum principle of Theorem 2.2 in [14] shows that $\varphi\equiv 0$ or $\varphi>0$. Thus we only need to deal with $\varphi>0$. In this situation, consider the conformal metric $g=\varphi^{\frac{4}{n-4}}g_0$. Then the Q-curvature satisfies

$$Q_g=\varphi^{-\frac{n+4}{n-4}}f(\varphi)\geq 0.$$

Making use of this fact, Lemma 2.5 shows that $R_g>0$. Then setting $u=\varphi^{\frac{2}{n-4}}$, one has

$$\langle\nabla Q_g,\nabla u\rangle_g=\frac{4}{n-4}\varphi^{\frac{4}{n-4}-1}F'(\varphi)|\nabla\varphi|_g^2\leq 0, \tag{4.3}$$

where $F(t)=t^{-\frac{n+4}{n-4}}f(t)$. Corollary 2.2 and the inequality yield that

$$0\geq\frac{n^2-2}{2(n-1)}\int_M u|E_g|_g^2R_g\mathrm{d}v_g+R_0\int_M|E_g|_g^2u^{-1}\mathrm{d}v_g.$$

Since $R_g>0$ and $R_0>0$, one must have $E_g=0$ which means that R_g is a constant. On one hand, if (M^n,g_0) is not conformally equivalent to the round sphere, Obata's theorem implies that φ must be a constant. On the other hand, if (M^n,g_0) is conformally equivalent to the round sphere, Corollary 2.2 and the identity (4.3) show that

$$\frac{4}{n-4}\int_M\varphi^{\frac{4}{n-4}-1}F'(\varphi)|\nabla\varphi|_g^2\mathrm{d}v_g\geq 0.$$

Since $F'(t)<0$ for all $t>0$ and $\varphi>0$, one must have $|\nabla\varphi|_g\equiv 0$ which means that φ is constant.

For $n=3$, one has

$$P_{g_0}\varphi\leq 0.$$

Proposition 2.1 in [17] shows that $\varphi\equiv 0$ or $\varphi>0$. Since the rest of the proof is the same as before, we omit the details.

For $n=4$, consider the conformal metric $g=e^{2\varphi}g_0$ and then

$$Q_g=e^{-4\varphi}f(e^{\varphi})\geq 0.$$

Immediately, Lemma 2.5 yields that $R_g>0$. The remaining is the same as the proof of Theorem 1.9 and we omit it. So the proof is finished.

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