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Calculus of Variations

A flow approach to mean field equation

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Abstract
The current article aims to study the existence of the solutions for the mean field Eq. (1.1) on a closed Riemann surface (M, g). To serve this, one constructs a non-local gradient-like flow and studies its properties. As the consequence, one can conclude the existence of the solutions without the crucial assumption on a given coefficient h.

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1 Introduction

Let (M, g) be a closed Rieman surface with a given metric g. For simplicity, the volume of M is assumed to be 1 throughout the whole work. Let h be a smooth, somewhere positive function on M. Denoted by ρ a fixed number 8π. The current paper mainly concerns with the following so-called mean field Eq.

$$-\Delta_g u = \rho \left(\frac{h(x)e^u}{\int_M h(x)e^u \mathrm{d}\mu_g} - 1 \right), \tag{1.1}$$

where Δ_g is the Laplace-Beltrami operator associated to metric g. It is well known that the solutions to (1.1) are the critical points of the following energy functional:

$$I(u) = \int_M \left(\frac{1}{2}|\nabla_g u|_g^2 + \rho u \right) \mathrm{d}\mu_g - \rho \ln \left(\int_M he^u \mathrm{d}\mu_g \right).$$

In seventies of the last century, by releasing the geometric restriction, Kazdan and Warner [20] initiated the study of the semi-linear equation

$$-\Delta_g u = he^u - \rho \tag{1.2}$$

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where ρ is a constant and h is a given smooth function. In the case where M is the standard sphere $\mathbb{S}^2$ and $\rho = 8\pi$, this equation usually is known as the Nirenberg equation which has been well studied, for example in [7, 8, 12, 25, 28, 31], and many others. In general situation, if $\rho < 0$ and $h \leq 0$, the existence of the solution to (1.2) is well known through the pioneering work of Kazdan and Warner [20]. When $\rho = 0$, the solvability of the Eq. (1.2) is perfectly settled [20]: it has a solution if and only if h changes sign and $\int_M h \mathrm{d}\mu_g < 0$. The case remains open when $\rho < 0$ and h changes sign. The possibility of blowing-up of the solutions made the problem more delicate. Interesting readers are referred to the references [1, 13, 20].

Our concern in this work is the mean field Eq. (1.1) which arises in various contexts such as the abelian Chern–Simons–Higgs models (see [5, 30, 32]). Notice that the Eq. 1.1 is the special case of (in fact, is equivalent to) the following equation:

$$- \Delta_g u = \rho \frac{e^u}{\int_M e^u \mathrm{d}\mu_g} - Q \tag{1.3}$$

for smooth functions ρ and Q. Existence of solutions of (1.3) has been widely studied, for example [4, 11, 14, 15, 17, 23, 24], and many others.

Recently, the flow approach has been adopted to the study of the solvability for (1.1) or (1.3), for example, [6, 12, 22, 27–29]. It seems this method is quite powerful.

Let us first mention two related results. The pioneering work on the Eq. (1.1) has been done in [16] by the variational method with the strong assumption that $h > 0$. Recently, in [33], the authors have extended the result to the case that h is just nonnegative. Again, the same conclusion as in [33] has been reconfirmed through flow method by Sun-Zhu in [29]. Motived by our recent work [12], we wish to revisit the existence of (1.1), hopefully get rid of the non-negativity assumption. The key advantage of [12] is that h being nonnegative is not needed. Our key observation is that, with $\rho = 8\pi$, if the solution blows up along the flow, it must blow up at only one point x_0 with $h(x_0) > 0$. To state our main theorem, some notations are needed. The Green function $G(x, p)$ on M is the function that satisfies the equation

$$\begin{cases} -\Delta_g G(x, p) = 8\pi\delta_p(x) - 8\pi, \\ \int_M G \mathrm{d}\mu_g = 0. \end{cases}$$

In normal coordinate system around p, G has expansion

$$\begin{aligned} G(x, p) = &- 4 \ln r + A(p) + b_1 x_1 + b_2 x_2 \\ &+ c_1 x_1^2 + 2c_2 x_1 x_2 + c_3 x_2^2 + O(r^3) \end{aligned} \tag{1.4}$$

where $r = \mathrm{dist}_g(x, p)$. We denote by M^+ the set $\{x \in M | h(x) > 0\}$. Now we are in position to state our main theorem:

Theorem 1.1 *Let (M, g) be a closed Riemann surface, $K(x)$ be its Gauss curvature. Suppose $h(x) \in C^\infty(M)$ with $\max_M h > 0$. For $\rho = 8\pi$, if the function $h(p)e^{A(p)/2}$ has a maximum point p_0 in M^+, with the property that*

$$\Delta_g \ln h(p_0) + 8\pi - 2K(p_0) > 0, \tag{1.5}$$

then the Eq. (1.1) has a smooth solution.

Remark 1.2 It costs nothing to point out that our method provides a more general result than the one stated here.

This paper is organized as follows. In Sect. 2, one should construct the aforementioned flow and study the decreasing property of the energy functional along the flow. In Sect. 3, we discuss the short, long time existence and its Hölder regularity for the flow we set up in Sect. 2. Section 4 will be contributed to the delicate analysis, drawing the conclusion that the blow up point can only occur at the inside point of M^+. Finally, Sect. 5 will be devoted to the necessary argument to achieve our main theorem.

2 The flow

Let $h(x)$ be a given smooth function, which is positive somewhere on M. Denoted by $\mathcal{M}$ the set of functions $\{u \in H^2(M) | \int_M he^u \mathrm{d}\mu_g = 1\}$. Our assumption on h simply implies that $\mathcal{M}$ is non-empty. Since our method works for $0 < \rho < 8\pi$ too, we write it in a general form with $0 < \rho \leq 8\pi$. The flow to be considered in this work can be stated as follows:

$$\partial_t u = \alpha(t)h - R \tag{2.1}$$

where the scalar curvature type function R is given by the formula

$$R = \frac{1}{\rho} e^{-u}(-\Delta_g u + \rho) \tag{2.2}$$

and the normalized coefficient $\alpha(t)$ is chosen so that

$$\alpha(t) = \frac{\int_M hRe^u \mathrm{d}\mu_g}{\int_M h^2 e^u \mathrm{d}\mu_g} \tag{2.3}$$

with the initial datum $u(0) = u_0 \in \mathcal{M}$. Observe that the flow Eqs. (2.1) and (2.3) imply

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_M he^u \mathrm{d}\mu_g = \int_M (\alpha h - R)he^u \mathrm{d}\mu_g = 0.$$

As a consequence of the choice of initial datum, there holds

$$\int_M he^{u(t)} \mathrm{d}\mu_g = \int_M he^{u(0)} \mathrm{d}\mu_g = 1 \tag{2.4}$$

for all t whenever the flow exists. With this at hand, the energy functional reduces to

$$J(u) := \int_M \left(\frac{1}{2} |\nabla_g u|_g^2 + \rho u \right) \mathrm{d}\mu_g. \tag{2.5}$$

Therefore our first observation can be stated as a Lemma:

Lemma 2.1 *On any finite time interval where the solution to the flow exists, there holds*

$$\frac{\mathrm{d}}{\mathrm{d}t} J(u) = -\rho \int_M (\alpha h - R)^2 e^u \mathrm{d}\mu_g.$$

Proof Making use of Eqs. (2.1) and (2.3), a direct computation yields

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}J(u) &= \int_M \langle \nabla_g u, \nabla_g u_t\rangle_g + \rho u_t \mathrm{d}\mu_g \\ &= \int_M (-\Delta_g u + \rho)u_t \mathrm{d}\mu_g \\ &= \rho \int_M R(\alpha h - R)e^u \mathrm{d}\mu_g \\ &= -\rho \int_M (\alpha h - R)^2 e^u \mathrm{d}\mu_g.\end{aligned}$$

Clearly Lemma follows from this. □

As a direct consequence of Lemma 2.1, $J(u)$ is non-increasing for $t \geq 0$. In particular, for any finite T if the flow exits on $[0, T)$, there holds

$$J(u(T)) + \rho \int_0^T \int_M (\alpha h - R)^2 e^u \mathrm{d}\mu_g \mathrm{d}t = J(u(0)). \tag{2.6}$$

Now rewrite the celebrated Moser-Trudinger inequality([18],Theorem 1.7) to get,

$$\begin{aligned}\ln \int_M e^u \mathrm{d}\mu_g &\leq \frac{1}{16\pi}\int_M |\nabla_g u|_g^2 \mathrm{d}\mu_g + \int_M u \mathrm{d}\mu_g + c \\ &= \frac{1}{8\pi}J(u) + \frac{8\pi - \rho}{8\pi}\int_M u \mathrm{d}\mu_g + c,\end{aligned} \tag{2.7}$$

for all $u \in H^1(M)$ with c a constant depending only on (M, g). Applying the Jensen's inequality to the second term on the right and moving the resulting term to left, we conclude

$$\frac{\rho}{8\pi}\ln \int_M e^u \mathrm{d}\mu_g \leq \frac{1}{8\pi}J(u) + c. \tag{2.8}$$

This, together with Lemma 2.1, implies

$$e^{\int_M u(t)\mathrm{d}\mu_g} \leq \int_M e^{u(t)} \mathrm{d}\mu_g \leq C \exp\left(\frac{1}{\rho}J(u_0)\right). \tag{2.9}$$

Here and thereafter, we denote by C a constant which may be different from line to line. Inequality (2.9) provides the upper bound for $\bar{u}$. On the other hand, we will have

$$\max h \int_M e^{u(t)} \mathrm{d}\mu_g \geq \int_M h e^{u(t)} \mathrm{d}\mu_g = 1. \tag{2.10}$$

Combing (2.8) and (2.10), there holds

$$J(u(t)) \geq -\rho \ln(\max h) + C. \tag{2.11}$$

If the flow Eq. (2.1) globally exists in time, then by letting $T \to \infty$ in the Eq. (2.6), we obtain

$$\int_0^\infty \int_M (\alpha h - R)^2 e^{u(t)} \mathrm{d}\mu_g \mathrm{d}t \leq \ln(\max h) + C + \frac{1}{\rho}J(u_0). \tag{2.12}$$

3 Short and long time existence

The short time existence follows from the standard parabolic theory. In fact, Brendle [2] proved the short time existence for higher order conformal flow and more general theory is given by Huisken and Polden [19]. The more detailed argument can be found in the appendix of [12].

Now, we discuss the global existence in time. As a first step, we state and prove the following lemma.

Lemma 3.1 *For any $T > 0$ and $p \geq 1$, any smooth sloution of the flow (2.1) which exists on $[0, T)$ satifies*

$$\sup_{0 \leq t < T} \int_M e^{pu(t)} \mathrm{d}\mu_g \leq C(p, T, \|u_0\|_{H^1(M)}).$$

Proof By the normalization (2.4) and Eq. (2.2), there holds

$$\alpha - 1 = \int_M (\alpha h - R) e^u \mathrm{d}\mu_g.$$

Thus the Hölder inequality and the estimate (2.9) yield the inequality

$$(\alpha - 1)^2 \leq \int_M e^u \mathrm{d}\mu_g \int_M (\alpha h - R)^2 e^u \mathrm{d}\mu_g \leq C\|u_0\|_{H^1(M)} \int_M (\alpha h - R)^2 e^u \mathrm{d}\mu_g. \quad (3.1)$$

Now a rather easy calculation yields

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \int_M e^{pu(t)} \mathrm{d}\mu_g = p \int_M e^{pu(t)} (\alpha h - R) \mathrm{d}\mu_g \\
&= \alpha p \int_M h e^{pu(t)} \mathrm{d}\mu_g - \frac{p}{\rho} \int_M e^{(p-1)u(t)} (-\Delta_g u + \rho) \mathrm{d}\mu_g \\
&= -\frac{p(p-1)}{\rho} \int_M e^{(p-1)u(t)} |\nabla_g u|_g^2 \mathrm{d}\mu_g \\
&\quad - p \int_M e^{(p-1)u(t)} \mathrm{d}\mu_g + \alpha p \int_M h e^{pu(t)} \mathrm{d}\mu_g \\
&\leq 2p \max |h| ((\alpha - 1)^2 + 1) \int_M e^{pu(t)} \mathrm{d}\mu_g,
\end{aligned}$$

by observing that $|\alpha| \leq |\alpha - 1| + 1 \leq 2\left((\alpha - 1)^2 + 1\right)$. By making use fo the estimates (2.12) and (3.1), we divide above inequality both sides by $\int_M e^{pu} \mathrm{d}\mu_g$ and integrate the resulting inequality to get

$$\begin{aligned}
\ln \int_M e^{pu(t)} \mathrm{d}\mu_g - \ln \int_M e^{pu(0)} \mathrm{d}\mu_g &\leq C \left(\int_0^t \int_M (\alpha h - R)^2 e^u \mathrm{d}\mu_g \mathrm{d}t + t \right) \\
&\leq C(p, T, \|u_0\|_{H^1(M)}).
\end{aligned}$$

Note that Moser inequality

$$\ln \int_M e^{pu(0)} \mathrm{d}\mu_g \leq \frac{p^2}{16\pi} \int_M |\nabla_g u(0)|_g^2 \mathrm{d}\mu_g + p \int_M u(0) \mathrm{d}\mu_g + c \leq C(p)(\|u(0)\|_{H^1(M)}^2 + 1),$$

implies

$$\sup_{0 \leq t < T} \int_M e^{pu(t)} \mathrm{d}\mu_g \leq C\left(p, T, \|u_0\|_{H^1(M)}\right). \quad (3.2)$$

This completes the proof of Lemma. □

With Lemma 3.1 at hand, we are in position to get the H^2 norm bound for $u(t)$.

Lemma 3.2 *For any $T > 0$ with the property that the flow exists on $[0, T)$, there exists a positive constant $C(T, \|u_0\|_{H^2(M)})$ such that the H^2- norm of u is uniformly controlled by C, namely:*

$$\sup_{0\le t<T} \|u(t)\|_{H^2(M)} \le C\left(T, \|u_0\|_{H^2(M)}\right).$$

Proof First of all, the estimates (2.10) and (2.9), for simplicity, can be rewritten as

$$c_1 \le \int_M e^{u(t)}\mathrm{d}\mu_g \le c_2 \tag{3.3}$$

where c_1, c_2 are constants which depend on $\max h$, $\|u_0\|_{H^1(M,g)}$. To make the notation simple, in the following estimates, the constant $C(T)$ will be allowed to depend on the function $h(x)$ as well as the initial datum $\|u_0\|_{H^2(M)}$ which may be different from line to line. Set, now,

$$A(t) = \{x \in M | e^{u(t)} \ge \frac{1}{2}c_1\} \ne \emptyset.$$

Thus Lemma 3.1 can be used to get the following estimate:

$$\begin{aligned} c_1 \le \int_M e^{u(t)}\mathrm{d}\mu_g &= \int_{M\setminus A(t)} e^{u(t)}\mathrm{d}\mu_g + \int_{A(t)} e^{u(t)}\mathrm{d}\mu_g \\ &\le \frac{1}{2}c_1 \int_M \mathrm{d}\mu_g + \left(\int_M e^{2u(t)}\mathrm{d}\mu_g\right)^{\frac{1}{2}} |A(t)|_g^{\frac{1}{2}} \end{aligned}$$

where $|A(t)|_g$ stands for the area of $A(t)$. This provides the control on the size of the set $A(t)$:

$$|A(t)|^{\frac{1}{2}} \ge C(T)^{-1}\frac{1}{2}c_1.$$

Therefore, by the definition of $A(t)$ and Jensen's inequality, there holds

$$\left(\ln\frac{c_1}{2}\right)|A(t)|_g \le \int_{A(t)} u(t)\mathrm{d}\mu_g \le |A(t)|_g \ln\left(|A(t)|_g^{-1}\int_{A(t)} e^{u(t)}\mathrm{d}\mu_g\right),$$

which, in turn, implies the estimate on the average of u

$$\begin{aligned} |\bar{u}| :=& |\int_M u(t)\mathrm{d}\mu_g| \\ \le& |\int_{M\setminus A(t)} u(t)\mathrm{d}\mu_g| + |\int_{A(t)} u(t)\mathrm{d}\mu_g| \\ \le& |M\setminus A(t)|_g^{\frac{1}{2}}\left(\int_{M\setminus A(t)} u^2(t)\mathrm{d}\mu_g\right)^{\frac{1}{2}} + C(T) \\ \le& \sqrt{1 - C(T)^{-1}}\|u(t)\|_{L^2(M)} + C(T). \end{aligned}$$

Thus, combine with Poincaré inequality to get

$$\begin{aligned} \|u(t)\|_{L^2(M)} &\le c\|\nabla_g u(t)\|_{L^2(M)} + |\bar{u}| \\ &\le c\|\nabla_g u(t)\|_{L^2(M)} + \sqrt{1 - C(T)^{-1}}\|u(t)\|_{L^2(M)} + C(T). \end{aligned}$$

After some simplifications, it can be rewritten as

$$\|u(t)\|_{L^2(M)} \leq C(T)\left(1+\|\nabla_g u(t)\|_{L^2(M)}\right). \tag{3.4}$$

On the other hand, with help of Young's inequality, one obtains

$$J(u_0) \geq J(u) \geq \frac{1}{2}\int_M |\nabla_g u(t)|_g^2 \mathrm{d}\mu_g - \epsilon \int_M u(t)^2 \mathrm{d}\mu_g - C_\epsilon. \tag{3.5}$$

Now, choose ϵ small, together with (3.4), to conclude that

$$\|u(t)\|_{H^1(M)} \leq C(T). \tag{3.6}$$

To get higher order derivative estimates, we first observe that

$$\begin{aligned}
\alpha \int_M h^2 e^{u(t)} \mathrm{d}\mu_g &= \int_M R h e^{u(t)} \mathrm{d}\mu_g \\
&= \frac{1}{\rho}\int_M h(-\Delta_g u + \rho)\mathrm{d}\mu_g \\
&= \frac{1}{\rho}\int_M \langle \nabla_g h, \nabla_g u\rangle_g \mathrm{d}\mu_g + \int_M h \mathrm{d}\mu_g.
\end{aligned}$$

The estimates (2.4) and (2.9) reveal the following estimate

$$1 = \left(\int_M h e^{u(t)} \mathrm{d}\mu_g\right)^2 \leq \int_M e^{u(t)} \mathrm{d}\mu_g \int_M h^2 e^{u(t)} \mathrm{d}\mu_g \leq C \int_M h^2 e^{u(t)} \mathrm{d}\mu_g. \tag{3.7}$$

Combine above two estimates, together with Hölder inequality and the estimate (3.6), to get

$$|\alpha| \leq C(T). \tag{3.8}$$

Now, for convenience, set $w(t) = e^{\frac{u(t)}{2}} \partial_t u$. Use (2.1) and (2.2) to rewrite them as

$$\Delta_g u = \rho\left(e^{\frac{u}{2}} w + 1 - \alpha h e^u\right). \tag{3.9}$$

Thus, it follows from this and a direct calculation that

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\Delta_g u|^2 \mathrm{d}\mu_g =& \int_M (\Delta_g u)\Delta_g(e^{-\frac{u}{2}} w)\mathrm{d}\mu_g \\
=& \rho \int_M (e^{\frac{u}{2}} w + 1 - \alpha h e^u)\Delta_g(e^{-\frac{u}{2}} w)\mathrm{d}\mu_g \\
=& -\rho \int_M |\nabla_g w|_g^2 \mathrm{d}\mu_g + \frac{\rho}{4}\int_M |\nabla_g u|_g^2 w^2 \mathrm{d}\mu_g \\
&+ \rho\alpha \int_M \langle \nabla_g(h e^u), \nabla_g(e^{-\frac{u}{2}} w)\rangle_g \mathrm{d}\mu_g.
\end{aligned}$$

To treat the last term, by Eqs. (3.8),(3.6) and Lemma 3.1 , it can proceed as follows:

$$\begin{aligned}
&|\alpha \int_M \langle \nabla_g(h e^u), \nabla_g(e^{-\frac{u}{2}} w)\rangle_g \mathrm{d}\mu_g| \\
&\quad = |\alpha \int_M e^{\frac{u}{2}}\left(h\langle \nabla_g u, \nabla_g w\rangle_g + \langle \nabla_g h, \nabla_g w\rangle_g - \frac{w}{2} h |\nabla_g u|_g^2 - \frac{w}{2}\langle \nabla_g h, \nabla_g u\rangle_g\right)\mathrm{d}\mu_g| \\
&\quad \leq \frac{1}{2}\int_M |\nabla_g w|_g^2 \mathrm{d}\mu_g + C(T)\left(1+\|\nabla_g u\|_{L^4(M)}^2\right) + \frac{1}{4}\int_M |\nabla_g u|_g^2 w^2 \mathrm{d}\mu_g.
\end{aligned}$$

Thus we conclude that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\Delta_g u|^2 \mathrm{d}\mu_g \leq -\rho \int_M |\nabla_g w|_g^2 \mathrm{d}\mu_g + \rho \int_M |\nabla_g u|_g^2 w^2 \mathrm{d}\mu_g$$
$$+C(T)\left(1+\|\nabla_g u\|^2_{L^4(M)}\right). \tag{3.10}$$

In order to handle the middle term on the right, we should use the well known interpolation inequality:

$$\|f\|^2_{L^4(M)} \leq c\|f\|_{L^2(M)}\|f\|_{H^1(M)}. \tag{3.11}$$

which holds for all $f \in H^1(M)$. More precisely, we have the estimates, together with (3.6),

$$\begin{aligned}
&\int_M w^2|\nabla_g u|_g^2 \mathrm{d}\mu_g \\
&\quad \leq \|w\|^2_{L^4(M)}\|\nabla_g u\|^2_{L^4(M)} \\
&\quad \leq c\|w\|_{L^2(M)}\|w\|_{H^1(M)}\|u\|_{H^1(M)}\|u\|_{H^2(M)} \\
\leq &\quad C(T)\|w\|^2_{L^2(M)}(\|u\|^2_{H^2(M)}+1)+\frac{1}{2}\|\nabla_g w\|^2_{L^2(M)}.
\end{aligned}$$

To deal with the last term of (3.10) on the right, we use (3.11) to get

$$\|\nabla_g u\|^2_{L^4(M)} \leq c\|u\|_{H^1(M)}\|u\|_{H^2(M)} \leq C(T)\|u\|_{H^2(M)}.$$

By the equivalence of the H^2 norm, one has

$$\|u\|^2_{H^2(M)} \leq C\left(\|\Delta_g u\|^2_{L^2(M)}+\|u\|^2_{L^2(M)}\right) \leq C(T)\left(\|\Delta_g u\|^2_{L^2(M)}+1\right). \tag{3.12}$$

Insert these estimates into (3.10) and simplify the resulting equation to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_M |\Delta_g u|^2 \mathrm{d}\mu_g \leq -\frac{\rho}{2}\int_M |\nabla_g w|_g^2 \mathrm{d}\mu_g + C(T)\left(1+\|w\|^2_{L^2(M)}\right)\left(\|\Delta_g u\|^2_{L^2(M)}+1\right),$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\ln(\|\Delta_g u\|^2_{L^2(M)}+1)\right) \leq C(T)(1+\|w\|^2_{L^2(M)}). \tag{3.13}$$

Notice that

$$\int_0^t \|w\|^2_{L^2(M)}\mathrm{d}t = \int_0^t \int_M e^u(\alpha h - R)^2 \mathrm{d}\mu_g \mathrm{d}t.$$

Therefore integrate (3.13) over the time interval $[0, T]$ to conclude that

$$\|\Delta_g u\|^2_{L^2(M)} \leq C(T).$$

Return to the estimate (3.12) to reach

$$\|u\|_{H^2(M)} \leq C(T).$$

This completes the proof. □

The last ingredient for global existence is the local Hölder estimate which follows from the previous lemma.

Proposition 3.3 *(Simon Brendle [3,Proposition 2.6]) Given any $T > 0$ and any $0 < \mu < 1$, there exists a constant $C(T)$ such that*

$$|u(x_1, t_1) - u(x_2, t_2)| \le C(T)((t_1 - t_2)^{\frac{\mu}{2}} + d(x_1, x_2)^{\mu})$$

for all $x_1, x_2 \in M$ and all $t_1, t_2 \in [0, T]$ satisfying $0 < t_1 - t_2 < 1$.

Proof It is not hard to imitate the proof of Proposition 2.6 in [3]. We omit the details for saving the length of the paper. □

With the help of Proposition 3.3, it is not hard to see that $u(T)$ exists so that the flow can continue to beyond T. Therefore the solution to the flow with given initial datum must exist globally.

4 Blow up analysis

When $\rho = 8\pi$, the flow may blow up in the sense that $\lim_{t\to\infty} \inf \int_M u(t)\mathrm{d}\mu_g = -\infty$. Our main purpose of the current section is to show that the flow develops at most one bubble which located at inside M^+. First of all, according to the Eq. (2.12), we have

$$\int_0^{\infty} \int_M u_t^2 e^u \mathrm{d}\mu_g \mathrm{d}t < +\infty.$$

It follows from this that there is a sequence of positive $\{t_n\}$ such that $t_n < t_{n+1}$ and $t_n \to \infty$ as $n \to \infty$

$$\lim_{n\to\infty} \int_M e^{u(t_n)} |\partial_t u(t_n)|^2 \mathrm{d}\mu_g = 0. \tag{4.1}$$

Set

$$u_n = u(t_n), \quad V_n = 8\pi\alpha(t_n)h, \quad f_n = 8\pi e^{\frac{u_n}{2}} \partial_t u(t_n). \tag{4.2}$$

Recall the flow equation again to have

$$-\Delta_g u_n + 8\pi = V_n e^{u_n} - f_n e^{\frac{u_n}{2}}. \tag{4.3}$$

Due to the estimates (3.1) and (4.1), one has

$$\lim_{n\to\infty} \alpha(t_n) = 1.$$

Thus, two basic facts are in order: on one hand, one has

$$|V_n| \le C, \quad \lim_{n\to\infty} \|f_n\|_{L^2(M)} = 0; \tag{4.4}$$

On the other hand, the estimate (2.9) implies

$$\{V_n e^{u_n}\} \in L^1(M). \tag{4.5}$$

Noitce that Hölder's inequality implies the estimate

$$\int_M |f_n e^{\frac{u_n}{2}}| \mathrm{d}\mu_g \le \left(\int_M f_n^2 \mathrm{d}\mu_g\right)^{\frac{1}{2}} \left(\int_M e^{u_n} \mathrm{d}\mu_g\right)^{\frac{1}{2}},$$

which, in turn, should be used to conclude that

$$\lim_{n\to\infty} \|f_n e^{\frac{u_n}{2}}\|_{L^1(M)} = 0. \tag{4.6}$$

Now we say a sequence $\{u_n\}$ satisfying (4.3) and (4.4) is a blow up sequence if

$$\lim_{n\to\infty}\inf\int_M u_n \mathrm{d}\mu_g = -\infty.$$

Lemma 4.1 *If $\{u_n\}$ is not a blow up sequence, then $\{u_n\}$ is bounded in $H^2(M,g)$.*

Proof The statement is rather standard. We provide a brief proof here. Since $\int_M u_n \mathrm{d}\mu_g \geq -C$, the volume upper bound (2.9) gives the control:

$$|\bar{u}_n| \leq C. \tag{4.7}$$

This bound on the average, together with the inequality (2.7), reveals a bound for its L^p norm for e^{u_n} for any positive p:

$$\ln\int_M e^{pu_n}\mathrm{d}\mu_g \leq \frac{p^2}{16\pi}\int_M |\nabla u_n|_g^2 \mathrm{d}\mu_g + p\int_M u_n \mathrm{d}\mu_g + c \leq \frac{p^2}{8\pi}J(u_0) + C. \tag{4.8}$$

This L^p norm estimate, together with point-wise boundedness, shows that $V_n e^{u_n}$ is uniformly bounded in L^p norm too. Making use of this bound, the Eq. (4.3) and triangle inequality for L^p norm, it follows that

$$\begin{aligned}\|\Delta_g u_n\|_{L^{\frac{3}{2}}(M)} \leq & \|V_n e^{u_n}\|_{L^{\frac{3}{2}}(M)} + \|f_n e^{\frac{u_n}{2}}\|_{L^{\frac{3}{2}}(M)} + 8\pi \\ \leq & \left(\int_M f_n^2 \mathrm{d}\mu_g\right)^{\frac{1}{2}}\left(\int_M e^{3u_n}\mathrm{d}\mu_g\right)^{\frac{1}{6}} + C \\ \leq & C,\end{aligned}$$

where the second inequality follows from the Hölder inequality.

By standard elliptic theory and (4.7), we have $\|u_n\|_{L^\infty(M)} \leq C\|\Delta_g u_n\|_{L^{\frac{3}{2}}(M)} + |\bar{u}_n| \leq C$. Then the convexity of L^2 norm implies

$$\begin{aligned}\|\Delta_g u_n\|_{L^2(M)} \leq & \|f_n e^{u_n/2}\|_{L^2(M)} + C \\ \leq & e^{\|u_n\|_{L^\infty(M)}/2}\|f_n\|_{L^2(M)} + C \\ \leq & C.\end{aligned}$$

Thus this estimate shows that $\|u_n\|_{H^2(M,g)}$ is uniformly bounded by some constant $C > 0$ and the proof of Lemma is complete. □

From now on, we assume $\{u_n\}$ is a blow up sequence. First of all, by the basic property of Green's function, it is rather easy to see that there exists a $C > 0$ such that

$$\|u_n - \bar{u}_n\|_{L^1(M)} \leq C. \tag{4.9}$$

However this bound is not enough to serve our purpose. A stronger estimate is needed. In order to achieve this, let us start with Brezis-Merle's well known estimate ([4], Theorem 1):

Lemma 4.2 *(see also [16]) Let $\Omega \subset M$ be a smooth domain. Assume u is a solution to*

$$\begin{cases} -\Delta_g w = f, & in\ \Omega \\ w = 0, & on\ \partial\Omega \end{cases}$$

with the right hand side f being in $L^1(M)$. Then, for every $0 < \delta < 4\pi$, there is a constant C depending only on δ and Ω such that

$$\int_\Omega \exp\left(\frac{(4\pi-\delta)|w|}{\|f\|_{L^1(M)}}\right) \mathrm{d}\mu_g \le C.$$

We are now in position to state and prove our next result in current section.

Theorem 4.3 *Suppose a blow up sequence $\{u_n\}$ has both properties(4.3) and (4.4). Then there exists finitely many points $x_1, \dots, x_L$ such that for any $s > 0$ and $l \in \{1, \dots, L\}$, there holds, up to a subsequence,*

$$\lim_{n\to\infty} \inf \int_{B_s^g(x_l)} |V_n| e^{u_n} \mathrm{d}\mu_g \ge 4\pi.$$

Moreover, for any $x \in M \backslash \{x_1, \dots, x_L\}$, there is a geodesic ball $\bar{B}_r^g(x) \subset M\backslash\{x_1, \dots, x_L\}$ and a positive constant $C = C_x$ such that

$$\|u_n - \bar{u}_n\|_{L^\infty(B_r^g(x))} \le C.$$

Proof First of all, we claim that there exists at least one such point. Suppose the claim is not true. Then, for every $x \in M$, there exist constants $\delta = \delta_x \in (0, \pi)$, $r = r_x \in (0, \frac{\mathrm{inj}(M)}{4})$, $N_1 = N_x \in \mathbb{N}$ such that for a small ball around x, $B_{4r}^g(x) \subset M$, up to a subsequence,

$$\int_{B_{4r}^g(x)} |V_n e^{u_n}| \mathrm{d}\mu_g \le 4\pi - 3\delta, \quad \text{for any } n \ge N_1.$$

On the other side, the limit (4.6) reveals that there exists a large integer N_2 such that, for previous subsequence,

$$\int_{B_{4r}^g(x)} |f_n| e^{\frac{u_n}{2}} \mathrm{d}\mu_g \le \int_M |f_n| e^{\frac{u_n}{2}} \mathrm{d}\mu_g \le \delta, \quad \text{for any } n \ge N_2.$$

Set $N = \max\{N_1, N_2\}$. Then the those two estimates yield

$$\int_{B_{4r}^g(x)} |V_n e^{u_n} - f_n e^{\frac{u_n}{2}}| \mathrm{d}\mu_g \le 4\pi - 2\delta, \quad \text{for any } n \ge N. \tag{4.10}$$

Next, we need to constant an approximation solution. As first step, one solves the following Dirichlet boundary value problem on the ball to get the function y

$$\begin{cases} \Delta_g y = 8\pi & \text{in } B_{4r}^g(x) \\ y \quad = 0 & \text{on } \partial B_{4r}^g(x). \end{cases}$$

Standard elliptic theory reveals that y is bounded in $L^\infty(B_{4r}^g(x))$. For the second step, one solves a different Dirichlet boundary value problem to get a sequence of functions $\{w_n\}$ on the ball $B_{4r}^g(x)$,

$$\begin{cases} -\Delta_g w_n = V_n e^{u_n} - f_n e^{\frac{u_n}{2}} & \text{in } B_{4r}^g(x) \\ w_n = 0 & \text{on } \partial B_{4r}^g(x), \end{cases}$$

Apply Lemma 4.2 to w_n with $f = V_n e^{u_n} - f_n e^{\frac{u_n}{2}}$ on $\Omega = B_{4r}^g(x)$, together with aid of Hölder inequality, to conclude

$$\|e^{|w_n|}\|_{L^p(B_{4r}^g(x))} \le C, \quad p = \frac{4\pi-\delta}{4\pi - 2\delta} > 1,$$

just observe that the choice of p satisfies the required estimate: $p \le \frac{4\pi-\delta}{\int_\Omega |f| \mathrm{d}\mu_g}$. In particular, we know that $\{w_n\}$ is bounded in $L^1\left(B^g_{4r}(x)\right)$. Finally, we set an approximation solution $h_n := u_n - \bar{u}_n - y - w_n$. It is clear that h_n is harmonic in $B^g_{4r}(x)$. Hence, we conclude that

$$\begin{aligned}\|h_n\|_{L^\infty\left(B^g_{2r}(x)\right)} &\le C\|h_n\|_{L^1\left(B^g_{4r}(x)\right)}\\ &\le C\left(\|u_n - \bar{u}_n\|_{L^1(M)} + \|y\|_{L^1\left(B^g_{4r}(x)\right)} + \|w_n\|_{L^1\left(B^g_{4r}(x)\right)}\right)\\ &\le C,\end{aligned}$$

by our previous estimate (4.9). This, together with the upper bound of $\bar{u}_n$, implies $e^{u_n} \in L^p(B^g_r(x))$. Thus apply standard elliptic theory to the equation for w_n to conclude that $w_n \in L^\infty(B^g_r(x))$. Hence, upon covering M with finitely many balls $B^g_{r_i}(x_i)$, we obtain $u_n - \bar{u}_n \in L^\infty(M, g)$. Now the estimate (3.3) imples

$$0 < c_1 \le \int_M e^{u_n} \mathrm{d}\mu_g = e^{\bar{u}_n} \int_M e^{(u_n - \bar{u}_n)} \mathrm{d}\mu_g \le C e^{\bar{u}_n}.$$

A simple observation reveals that the right side of above estimate tends to 0 as $n \to \infty$ due to the definition of blow up assumption. We reach a desired contradiction. Therefore there exists $x_1 \in M$ and $\forall r_1 > 0$ such that

$$\liminf_{n\to\infty} \int_{B^g_{r_1}(x_1)} |V_n| e^{u_n} \ge 4\pi. \tag{4.11}$$

Thus our claim follows.

Next if $\int_{M\setminus B^g_{r_1}(x_1)} |V_n| e^{u_n} \mathrm{d}\mu_g < 4\pi - 3\delta$, then, as before, we can show that $\|u_n - \bar{u}_n\|_{L^\infty(M\setminus B^g_{\frac{r_1}{2}}(x_1))} \le C < +\infty$. This shows the first part of Theorem holds true with $l = 1$. If $\int_{M\setminus B^g_{r_1}(x_1)} |V_n| e^{u_n} \mathrm{d}\mu_g > 4\pi$, we claim again that there exist $x_2 \ne x_1$ and $\forall r_2 > 0$ such that $B^g_{r_2}(x_2) \cap B^g_{r_1}(x_1) = \emptyset$ and $\lim_{n\to\infty} \int_{B^g_{r_2}(x_2)} |V_n| e^{u_n} \ge 4\pi$, up to a subsequence. In fact, suppose $\forall x \in M\setminus B^g_{\frac{r_1}{2}}(x_1)$, there exists r_x such that $\int_{B^g_{r_x}(x)} |V_n| e^{u_n} \mathrm{d}\mu_g < 4\pi - 3\delta$, again finitely covering argument shows that $\int_{M\setminus B^g_{r_1}(x_1)} |V_n| e^{u_n} \mathrm{d}\mu_g < 4\pi - 3\delta$ which contradicts to our assumption. Then we either stop at $l = 2$ or we can repeat this procedure to get $x_3, x_4 \ldots$ etc. This iteration must terminate after finitely many steps. Indeed, having determined $x_1, \ldots, x_L$ as above, since the balls $B^g_{r_i}(x_i)$ are disjoint, we can bound

$$4\pi L \le \sum_{i=1}^L \int_{B^g_{r_i}(x_i)} |V_n| e^{u_n} \mathrm{d}\mu_g = \int_{\cup_{i=1}^L B^g_{r_i}(x_i)} |V_n| e^{u_n} \mathrm{d}\mu_g \le \int_M |V_n| e^{u_n} \mathrm{d}\mu_g \le C,$$

we find

$$4\pi L \le C,$$

which implies that L is finite. After L steps, according to our iterated step, we must have

$$\|u_n - \bar{u}_n\|_{L^\infty\left(B^g_r(x)\right)} \le C.$$

□

Our next task is to show that, for each blow-up point $x \in S := \{x_1 \ldots, x_L\}$, $h(x)$ must be positive. For later reference, we state it as a lemma.

Lemma 4.4 *For each $x \in S$, $h(x) > 0$. Furthermore, the following limit holds true:*

$$\lim_{r\to 0}\lim_{n\to\infty}\inf \int_{B_r^g(x_0)} he^{u_n}\mathrm{d}\mu_g \geq \frac{1}{2}.$$

Proof We should adopt the usual contradiction argument.

Firstly, suppose $h(x_0) = 0$ for some point $x_0 \in S$. Since S has only finitely many points, it is not hard to see that we can choose $r_1 > 0$ small such that $B_{2r_1}^g(x_0) \cap S = \{x_0\}$ and

$$|h(x)| \leq (16C_1)^{-1}$$

for all $x \in B_{r_1}^g(x_0)$ by continuity of the function h, where C_1 is the upper bound of $\int_M e^{u_n}\mathrm{d}\mu_g$ appeared in the estimate (2.9). With this at hand, we should easily get

$$\int_{B_{r_1}^g(x_0)} |V_n|e^{u_n}\mathrm{d}\mu_g \leq \frac{8\pi\alpha(t_n)\int_{B_{r_1}^g(x_0)} e^{u_n}\mathrm{d}\mu_g}{16C_1} \leq \pi,$$

for sufficiently large n, which contradicts to the estimate (4.11).

Secondly, assume $h(x_0) < 0$ for some $x_0 \in S$. Then there exists $r_0 > 0$ such that $B_{3r_0}^g(x_0) \cap S = \{x_0\}$ and $h(x) < 0$ for all $x \in B_{3r_0}^g(x_0)$. Now, we construct a cut-off function $0 \leq \eta(x) \leq 1$ with the property that $\eta(x) = 1$ in $B_{r_0}^g(x_0)$, $\eta(x) = 0$ in $M\backslash B_{2r_0}^g(x_0)$ and $|\nabla\eta|^2 + |\Delta_g\eta^2| \leq \frac{C}{r_0^2}$ in $B_{2r_0}^g(x_0)\backslash B_{r_0}^g(x_0)$. Multiply by $\eta^2 e^{u_n}$ on both sides of the Eq. (4.3) and integrate the resulting equation, with help of integration by parts, to get,

$$\begin{aligned}\int_M \left(4|\nabla(\eta e^{\frac{u_n}{2}})|^2 + e^{u_n}\Delta\eta^2 - 4e^{u_n}|\nabla\eta|^2\right)\mathrm{d}\mu_g + 8\pi\int_M \eta^2 e^{u_n}\mathrm{d}\mu_g \\ = \int_M \eta^2 V_n e^{2u_n} - \int_M f_n\eta^2 e^{\frac{3u_n}{2}}\mathrm{d}\mu_g.\end{aligned}$$

Notice that $\int_M \eta^2 V_n e^{2u_n}\mathrm{d}\mu_g \leq 0$ by assumption on h and $\|u_n - \bar{u}_n\|_{L^\infty(B_{2r_0}^g(x_0)\backslash B_{r_0}^g(x_0))} \leq C$ as shown in Theorem 4.3. By those facts, together with the upper bound of $\bar{u}_n$(see the estimate (2.9)), one obtains:

$$\begin{aligned}4\int_M |\nabla(\eta e^{\frac{u_n}{2}})|^2\mathrm{d}\mu_g &\leq \int_M |f_n|\eta^2 e^{\frac{3}{2}u_n}\mathrm{d}\mu_g + C \\ &\leq \int_{B_{r_0}^g} |f_n|e^{\frac{3u_n}{2}}\mathrm{d}\mu_g + \|f_n\|_{L^2(M)}\|e^{\frac{3u_n}{2}}\|_{L^2(B_{2r_0}^g(x_0)\backslash B_{r_0}^g(x_0))} + C \\ &\leq \int_{B_{r_0}^g} |f_n|e^{\frac{3u_n}{2}}\mathrm{d}\mu_g + e^{\frac{3\bar{u}_n}{2}}\|f_n\|_{L^2(M)}\|e^{\frac{3(u_n-\bar{u}_n)}{2}}\|_{L^2(B_{2r_0}^g(x_0)\backslash B_{r_0}^g(x_0))} + C \\ &\leq \int_M |f_n|(\eta e^{\frac{u_n}{2}})^3\mathrm{d}\mu_g + C \\ &\leq \|f_n\|_{L^2(M)}\|\eta e^{\frac{u_n}{2}}\|_{L^6(M)}^3 + C.\end{aligned}$$

Thus, making use of Gagliardo-Nirenberg inequality and the boundedness of $\int_M e^{u_n}\mathrm{d}\mu_g$, one sees that the following estimate holds true:

$$\|\eta e^{\frac{u_n}{2}}\|_{L^6(M)}^3 \leq C\|\nabla(\eta e^{\frac{u_n}{2}})\|_{L^2(M)}^2\|\eta e^{\frac{u_n}{2}}\|_{L^2(M)} \leq C_1\|\nabla(\eta e^{\frac{u_n}{2}})\|_{L^2(M)}^2.$$

Now, notice that, due to the fact that $\|f_n\|_{L^2(M)} \to 0$ as $n \to \infty$, we can select a sufficiently large number N such that for $n \geq N$, $C_1\|f_n\|_{L^2(M)} \leq 2$, which yields

$$\|\nabla(\eta e^{\frac{u_n}{2}})\|_{L^2(M)}^2 \leq C.$$

Therefore, by Sobolev embedding, for all $p > 1$, one has

$$\|e^{\frac{u_n}{2}}\|_{L^p(B^g_{r_0}(x_0))} \leq C\|e^{\frac{u_n}{2}}\|_{H^1(B^g_{r_0}(x_0))} \leq C.$$

From this, we conclude that $\|u_n\|_{L^\infty(B^g_{r_0}(x_0))} \leq C$ which contradicts with the fact that u_n blows up at x_0.

Thus our first claim follows.

Now notice that $\alpha(t) \to 1$, by Theorem 4.3, at any blow up point x_0, one reaches:

$$\lim_{r\to 0}\lim_{n\to\infty}\inf\int_{B^g_r(x_0)} he^{u_n}\mathrm{d}\mu_g \geq \frac{1}{2}.$$

This justifies our second claim and hence the proof of Lemma is complete. □

Theorem 4.5 *For any blow up sequence* $\{u_n\}$, *there holds*

$$\lim_{n\to\infty}\max_M u_n = +\infty.$$

Moreover, $\{u_n\}$ *converges to* $-\infty$ *uniformly on compact subsets of* $M\backslash S$ *and* $L = \sharp S = 1$.

Proof We argue with contradiction. Suppose there exists a subsequence of $\{u_n\}$ which satisfies the control

$$\lim_{n\to\infty}\max_M u_n \leq C < +\infty.$$

For simplicity, we still denote by $\{u_n\}$. Use this bound to obtain:

$$\begin{aligned}
\int_M(-\Delta_g u_n + \rho)^2\mathrm{d}\mu_g =& \int_M(\rho R_n e^{u_n})^2\mathrm{d}\mu_g\\
\leq& C\int_M R_n^2 e^{u_n}\mathrm{d}\mu_g\\
\leq& C[\int_M(R_n - \alpha(t_n)h)^2e^{u_n}\mathrm{d}\mu_g + \alpha(t_n)^2\int_M h^2e^{u_n}\mathrm{d}\mu_g]\\
\leq& C,
\end{aligned}$$

where we have used the estimate (4.1). Thus, standard elliptic estimate provides the bound:

$$\|u_n - \bar{u}_n\|_{L^\infty(M,g)} \leq C.$$

By the estimate (2.10), there holds

$$0 < (\max h)^{-1} \leq \int_M e^{u_n}\mathrm{d}\mu_g = \int_M e^{u_n-\bar{u}_n}\mathrm{d}\mu_g e^{\bar{u}_n} \leq Ce^{\bar{u}_n}$$

which contradicts with the fact that $\bar{u}_n \to -\infty$ as $n \to \infty$.

Now let $K \subset M\backslash S$ be a compact set. As a consequence, we have $d(K, S) := r_0 > 0$. Then for any $x \in K$, there exists a small number $r_x < \frac{r_0}{2}$ such that $B^g_{r_x}(x) \subset M\backslash S$ satisfies the property:

$$\int_{B^g_{r_x}(x)}|V_n|e^{u_n}\mathrm{d}\mu_g < 4\pi - 3\delta,$$

and

$$\int_{B^g_{r_x}(x)}|f_n|e^{\frac{u_n}{2}}\mathrm{d}\mu_g < \delta$$

for some constant $\delta > 0$. The property of the set S implies that $\|u_n - \bar{u}_n\|_{L^\infty(K)} \leq C$ which yields $u_n \to -\infty$ uniformly on K.

Then, by the Eq. (2.4) and Lemma 4.4, one has

$$\begin{aligned}
1 &= \lim_{n\to\infty} \int_M he^{u_n} \mathrm{d}\mu_g \\
&= \sum_{x\in S} \lim_{r\to 0} \lim_{n\to\infty} \int_{M\setminus B_r^g(x)} he^{u_n} \mathrm{d}\mu_g + \sum_{x\in S} \lim_{r\to 0} \lim_{n\to\infty} \int_{B_r^g(x)} he^{u_n} \mathrm{d}\mu_g \\
&= \sum_{x\in S} \lim_{r\to 0} \lim_{n\to\infty} \int_{B_r^g(x)} he^{u_n} \mathrm{d}\mu_g \\
&\geq \frac{1}{2} \sharp S.
\end{aligned}$$

On one hand, obviously, $\sharp S \geq 1$ since the sequence blows up. On the other hand, by previous estimate, in order to reach our conclusion, we only need to exclude the case $\sharp S = 2$. To achieve this, we again do contradiction argument. Mainly one may follow the blow up argument as appeared in the proof of Lemma 1 in [21] with aids of the classification theorem in [9]. However one should provide another proof by making use of the variance of Aubin's inequality(Theorem 2.1, [10]). For convenience, we now state the inequality: for two subsets Ω_1 and Ω_2 of M with $\mathrm{dist}(\Omega_1, \Omega_2) \geq \epsilon_0 > 0$ for some positive ϵ_0 and any $u \in H^1(M)$, if there exists $a_0 > 0$ such that

$$\frac{\int_{\Omega_1} e^u \mathrm{d}\mu_g}{\int_M e^u \mathrm{d}\mu_g} \geq a_0, \quad \frac{\int_{\Omega_2} e^u \mathrm{d}\mu_g}{\int_M e^u \mathrm{d}\mu_g} \geq a_0,$$

then for any $\epsilon > 0$, there exists a constant $C(\epsilon, \epsilon_0, a_0) > 0$ such that

$$\int_M e^u \mathrm{d}\mu_g \leq C(\epsilon_0, \epsilon, a_0) \exp\left((\frac{1}{32\pi} + \epsilon) \int_M |\nabla u|^2 \mathrm{d}\mu_g + \int_M u \mathrm{d}\mu_g \right). \tag{4.12}$$

Now, suppose $\sharp S = 2$. By Lemma 4.4, we assume $S = \{x_1, x_2\}$ and $a_1 = \max\{h(x_1), h(x_2)\} > 0$. We choose suitable $r_1 > 0$ such that $B_{2r_1}^g(x_1) \cap B_{2r_1}^g(x_2) = \emptyset$ and $0 < h(x) \leq 2a_1$ on $B_{2r_1}^g(x_1) \cup B_{2r_1}^g(x_2)$. Set $\Omega_i = B_{r_1}^g(x_i)$, $i = 1, 2$. For n sufficiently large, by Lemma 4.4, there holds

$$\int_{\Omega_i} 2a_1 e^{u_n} \mathrm{d}\mu_g \geq \int_{\Omega_i} he^{u_n} \mathrm{d}\mu_g \geq \frac{1}{4}.$$

Together with (2.9), there holds

$$\frac{\int_{\Omega_i} e^{u_n} \mathrm{d}\mu_g}{\int_M e^{u_n} \mathrm{d}\mu_g} \geq \frac{1}{8a_1 C \exp\left(\frac{1}{\rho} J(u_0)\right)}.$$

Choose $\epsilon = \frac{1}{64\pi}$ in the inequality (4.12), one has

$$\int_M e^{u_n} \mathrm{d}\mu_g \leq C \exp\left(\frac{3}{32\pi} J(u_n) + \frac{1}{4} \bar{u}_n \right) \leq C \exp\left(\frac{1}{4} \bar{u}_n \right).$$

The right side tends to 0 as $n \to \infty$ while left hand side has uniform positive lower bound by the inequality (2.10). This contradiction shows that $\sharp S = 2$ cannot be true. Therefore the proof of Theorem is complete. $\square$

Theorem 4.6 *If the flow* (2.1) *develops a singularity, then we have*

$$J(u(t)) \geq C_0 = -4\pi \max_{x\in M^+} (A(x) + 2\ln h(x)) - 8\pi \ln \pi - 8\pi, \forall t \geq 0$$

where $A(x)$ is the regular part of the Green function G as in (1.4).

Proof Since the flow we work with keeps the number $\int_M he^{u(t)}$ being always 1, the energy $J(u(t))$ equals the energy functional $I(u(t))$. By Lemma 4.4 and Theorem 4.5, we may assume there exists a subsequence u_n such that it blows up only at x_0. On one hand, we have

$$\begin{aligned} 1 =& \lim_{r\to 0} \lim_{n\to\infty} \int_{B_r^g(x_0)} he^{u_n} \mathrm{d}\mu_g \\ =& h(x_0) \lim_{r\to 0} \lim_{n\to\infty} \int_{B_r^g(x_0)} e^{u_n} \mathrm{d}\mu_g \\ =& h(x_0) \lim_{n\to\infty} \int_M e^{u_n} \mathrm{d}\mu_g. \end{aligned}$$

On the other hand, if we set $v(t) = u(t) - \ln \int_M e^{u(t)} \mathrm{d}\mu_g$ then it is easy to check the two facts that $I(u(t)) = I(v(t))$ and $\int_M e^{v(t)} \mathrm{d}\mu_g = 1$. Thus we reach the same estimate for a sequence $\{t_n\}$ as Thereom 1.2 in [29]

$$\lim_{n\to\infty} J(u(t_n)) = \lim_{n\to\infty} I(v(t_n)) \geq -4\pi (A(x_0) + 2\ln h(x_0)) - 8\pi \ln \pi - 8\pi.$$

Hence, by the monotonicity of $J(u(t))$, we reach at

$$J(u(t)) \geq C_0 = -4\pi \max_{x\in M^+} (A(x) + 2\ln h(x)) - 8\pi \ln \pi - 8\pi, \ \forall t \geq 0.$$

This completes the proof □

5 Global convergence

Proof of Theorem Notice that, if $\rho = 8\pi$, there is a unbounded time sequence $\{t_n\}$ such that $t_n < t_{n+1}$ and

$$\lim_{n\to\infty} \int_M e^{u(t_n)} |\partial_t u(t_n)|^2 \mathrm{d}\mu_g = 0. \tag{5.1}$$

By the lower bound of $J(u(t))$ along the flow as stated in Theorem 4.6, the existence of the solution to the mean field Eq. (1.1) is reduced to construct a smooth function under which J is strictly less than C_0. In fact, such kind of function was constructed in [16] under the assumption (1.5). Therefore, with this assumption, the flow can not blow up. By Lemma 4.1, there is a subsequence, still denote by $\{u_n\}$, of $\{u_n\}$ and $u_\infty \in H^2(M)$ such that $u_n := u(t_n) \to u_\infty$ weakly in $H^2(M, g)$ which implies strong convergence in $H^1(M, g)$ and at the same time, $\alpha(t_n) \to 1$ due to (3.1) and Lemma 5.1. It is well known that, Moser-Trudinger inequality reveals that $e^{u_n} \to e^{u_\infty}$ in $L^p(M, g)$ for any finite p. Take $t = t_n$ in the Eq. (2.1), i.e., in the equation

$$\rho e^u u_t = \rho\alpha he^u + \Delta_g u - \rho, \tag{5.2}$$

ans let n tend to ∞ to reach the conclusion that u_∞ satisfies the equation:

$$\Delta_g u_\infty + \rho he^{u_\infty} - \rho = 0. \tag{5.3}$$

Thus u_∞ is a weak solution we are looking for. By standard regularity theory, one then easily see that $u_\infty \in C^\infty(M)$ which is a classical solution to (1.1) we are search for. This completes the proof of our main theorem. □

Remark 5.1 As we might see, our main focus here is the existence of the solution to the mean field equation and does not care much about the convergence of the flow. Actually, with the help of the Lojasiewicz-Simon inequality (See Theorem 3 in [26]), we may show that $u(t)$ converges to u_∞ uniformly as $t \to \infty$. Moreover, we should be able to prove that the flow converges at least polynomially fast. Those facts will be appeared in our forthcoming work.

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