Discrete and Continuous Dynamical Systems

doi:10.3934/dcds.2023149

ASYMPTOTIC BEHAVIOR OF CONFORMAL METRICS ON TORUS

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(Communicated by Juncheng Wei)

Abstract. On flat torus with standard metric g_b, two metrics g_1 and g_2 with unit volume in the conformal class of g_b are said to be equivalent if they have the same Gaussian curvature, denoted by $g_1 \sim g_2$. The main purpose of this article was to study the boundary of the moduli space $[g_b]/\sim$. Our main conclusion was that as the energy minimizing metrics of the equivalent classes along a certain direction approach the boundary point, the corresponding metrics must be blown up at exactly one point, which sharpens Struwe's estimate (JEMS 2020).

1. **Introduction.** Let M be a torus with the standard flat metric g_b of unit volume. It is natural to consider the set of Gaussian curvature functions of all conformal metrics. This is the so-called prescribed Gaussian curvature problem on M, which is equivalent to solving the semi-linear equation on M

$$-\Delta_{g_b} u = fe^{2u}. \tag{1}$$

A classical remarkable work of Kazdan and Warner [17] shows that the equation (1) is solvable if, and only if, either $f \equiv 0$ or f changes sign and $\bar{f} := \int_M f \mathrm{d}\mu_{g_b} < 0$. Thus, for such a given nonzero function f, this result indicates that there exists at least one solution u to the equation (1). However, it might have multiple solutions that one does not know. Nevertheless, any solution to (1) induces a conformal metric $g = e^{2u} g_b$, which has the given Gaussian curvature f. In order to settle non-uniqueness, we introduce the new concept. We say two conformal metrics g_1 and g_2 in the conformal class $[g_b]$ are equivalent, denoted by $g_1 \sim g_2$, if their Gaussian curvature is the same. Let $\mathcal{M}$ be the set of equivalent classes, i.,e., $\mathcal{M} = [g_b]/\sim$. The natural question is how to describe the boundary of this moduli space.

Notice that by the aforementioned Kazdan-Warner theorem, a non-constant smooth function f_0 that satisfies $\sup_M f_0 = 0$ as a Gaussian curvature must lie on the boundary of above moduli space in some sense. Precisely, for each real number λ, the function $f_0 + \lambda$ can be realized as a Gaussian curvature in the conformal class $[g_b]$ if $\lambda \in (0, -\bar{f}_0)$. As λ tends to the endpoints of the interval, clearly the equivalent class of conformal metrics tends to the boundary of the moduli space $\mathcal{M}$. This is a description from geometrical point of view. Actually, from an analytical

2020 *Mathematics Subject Classification.* Primary: 35J20; Secondary: 58E30.

Key words and phrases. Prescribed Gaussian curvature, blow up analysis, asymptotic behavior.

The first and second authors are both supported by NSFC (No.12171231) and Xu's start-up research funds through Nanjing University with account No. 020314912205.

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point of view, the following classical equation called the Emden-Fowler equation or Gelfand's problem has been well studied

$$\begin{cases} -\Delta u &= \lambda e^u \quad \text{in } \Omega \\ \quad u &= 0 \qquad \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary and λ is a positive constant. The above equation also has physical background, which arises in the theory of thermonic emission, isothermal gas sphere, gas combustion and many others (see [13]). We refer to the introduction in [23] for a nice survey. In particular, when λ tends to zero from above, the minimal solutions tends to zero (see [24] for more details). On a closed Riemann surface, the situation becomes very different, just as Theorem 1.1 in the current paper. For non-minimal solutions of the above equation, Weston [25] and Moseley [20] have constructed a branch of non-minimal solutions on the simply connected domain, and their solutions make one-point blow-up as $\lambda \downarrow 0$. Moreover, the asymptotic behavior of the solutions as $\lambda \downarrow 0$ has been well studied and classified in [21] where the quantitative phenomenon occurs. Let us mention some well-known results such as [6] and [19] under general assumptions without Dirichlet condition, works [8] and [18] related to mean field equations on closed Riemann manifolds and many others. In the current paper, we will show that the minimal solutions share the similar bubble phenomenon on closed Riemann surface.

Gamlimberti [12], refined by Struwe [22], already provides a nice description for this limit. In fact, the construction of the equivalent class in their work can be interpolated as the radial directional limit. Therefore, from this point of view in the current paper, we consider a more general case, namely, we consider

$$f_\lambda := f_0 + \lambda\eta,$$

where η is a positive smooth function with $\int_M \eta \mathrm{d}\mu_{g_b} = 1$ over M. It is not hard to see for each $\lambda \in (0, -\bar{f}_0)$, we have

$$\sup_M f_\lambda > 0 \quad \text{and} \quad \int_M f_\lambda \mathrm{d}\mu_{g_b} < 0.$$

The equivalent class can be re-parameterized by the set of functions on M as follows:

$$\mathcal{M} = \{f \in C^\infty(M) | \sup_M f > 0, \int_M f \mathrm{d}\mu_{g_b} < 0\}.$$

In some sense, we can view f_0 as the boundary point for the set $\mathcal{M}$. Meanwhile, we can take η as an inward derivative of the moduli space $\mathcal{M}$ at the point f_0. Our main focus of this work is to study the boundary behavior of the equivalent metrics with prescribed Gaussian curvature f_λ. We hope that we can give a description of the boundary of the set $\mathcal{M}$.

First, for convenience, we set some notations; mainly the same as those in [22]. Set the Dirichlet energy

$$E(u) = \frac{1}{2}\int_M |\nabla u|^2 \mathrm{d}\mu_{g_b}.$$

Consider it as a functional on the hypersurface C_λ in $H^1(M, g_b)$, where

$$C_\lambda = \{u \in H^1(M, g_b) | \int_M f_\lambda e^{2u} \mathrm{d}\mu_{g_b} = 0\}.$$

Since the energy is invariant up to constant addition, we can consider the further normalized set

$$C_\lambda^* = \{u \in C_\lambda | \int \eta e^{2u} \mathrm{d}\mu_{g_b} = 1\}.$$

We define the infimum energy by

$$\beta_\lambda = \inf_{u \in C_\lambda} E(u) = \inf_{u \in C_\lambda^*} E(u).$$

Observe that, for each $\lambda \in (0, -\bar{f}_0)$, the minimizer of $E(u)$ in C_λ^* exists and is denoted by u_λ. It is well known that u_λ satisfies

$$-\Delta_{g_b} u_\lambda = \alpha_\lambda f_\lambda e^{2u_\lambda}, \tag{2}$$

with some positive constant α_λ defined by

$$\alpha_\lambda = \frac{\int_M 2|\nabla u_\lambda|^2 e^{-2u_\lambda} \mathrm{d}\mu_{g_b}}{-\int_M f_\lambda \mathrm{d}\mu_{g_b}} > 0.$$

Note that by the Kazdan-Warner theorem, we must have

$$\beta_\lambda \to \infty \quad \text{as } \lambda \downarrow 0. \tag{3}$$

With $\eta = 1$, Galimberti [12] showed that $\beta_\lambda \to \infty$ and the metric blows up at finitely many points, and they provided some possible characterizations of the bubbles when $\lambda \to 0$ (see Theorem 1.2 in [12]). Meanwhile, a complete description of the boundary point at $-\bar{f}_0$ has been given. It was shown that $\beta_\lambda \to 0$ and the collapse phenomenon of the corresponding metrics occurs. Furthermore, the other end as $\lambda \to 0$ is very delicate. Later, a more precise characterization in this case has been done by Struwe [22] . He mainly ruled out the "slow blow-up", which was allowed in Galimberti's theorem (See Theorem 1.1 in [12]). It is achieved by a Liouville-type result (Theorem 1.3 in [22]) plus an the observation that if f_0 has non-degenerate maximum points, the family metrics could only shape at most two spherical bubbles. In the process of estimating the upper bound of the number of the blow up points, it is controlled by the upper bound of the function $\beta_\lambda / \log(1/\lambda)$. In [12], the author does not have a precise upper bound, while in [22] the author shows that this is bounded by 4π, which can be used to conclude that the metrics blow up at two points at most. In the current paper, we will improve estimate Lemma 4.2 in [22] and give a sharper upper bound of $\beta_\lambda / \log(1/\lambda)$ in the more general case. With the help of this sharp estimate, we can show that the metrics will shape only one spherical bubble even in this general case. Instead of making use of the Liouville-type result as in [22] to rule out slow bubble, we take a new strategy based on a global version of Alexandrov-Bol's inequality (see Lemma 2.6), as well as our sharp energy estimate (see Theorem 2.4) under more general assumptions on f_0. The natural and interesting question is to study the same issue for higher genus surfaces, which has been done in [5],[11],[10] etc., and the asymptotic behavior on $\mathbb{R}^2$ has been tackled in [9],[8] etc.

Due to the limitation of our method, we need assume the zero set of f_0 has good behavior in certain sense. For readers' convenience, we first introduce a condition to describe the meaning of "in certain sense." We say f_0 is non-degenerate of order $2l$ at maximum points if at each maximum points of f_0 it has the expansion

$$f_0(x) = p_{2l}(x) + O(|x|^{2l+1}),$$

where p_{2l} is the homogeneous polynomial of degree $2l$ and $p_{2l}(x) < 0$ for all $x \neq 0$ and $l \geq 1$.

Now, we are ready to state our first result.

Theorem 1.1. *Let $f_0 \leq 0$ be the smooth, non-constant function with non-degenerate of order $2l$ with $l \geq 1$ at maximum points. Then, for suitable $\lambda_k \downarrow 0$, denote u_{λ_k} by u_k and suitable r_k, satisfying $\frac{r_k^{2l}}{\lambda_k} \to 0$ and points $p_k \to p_\infty \in M$ with $f_0(p_\infty) = 0$ as $k \to \infty$, and the following holds:*

(i) $u_k \to -\infty$ is locally uniformly on compact subdomains of $M_\infty := M \backslash \{p_\infty\}$.
(ii) We have in local Euclidean coordinates x around $p_k = 0$,

$$\tilde{u}_k(x) := u_k(r_k x) + \log r_k \to \tilde{u}_\infty(x) = \log\left(\frac{4}{4 + 4\pi\eta(p_\infty)|x|^2}\right)$$

in $H^2_{loc}(\mathbb{R}^2)$.

In the course of proof, our method works for more general situations. Before we state it, we introduce the uniform cone condition (denoted as (**UCC**) for simplicity) as follows: For given $0 < \theta_0 < \pi/4$, $d_0 > 0$, the cone, denoted by C_0, is defined by

$$C_0 = \{x = (x_1, x_2) \in \mathbb{R}^2 | 0 < x_2 < \tan(\theta_0)x_1, |x| < d_0\}.$$

We say f_0 has the UCC if, for any $p \in M$ with $\mathrm{dist}(p, M_0) < d_0$ where $M_0 = \{p \in M | f_0(p) = 0\}$, there is a rotated copy $C_p \subset \mathbb{R}^2$ of C_0 and a constant $A_0 > 0$ independent of p such that, in the Euclidean coordinates x around $p = 0$,

$$A_0 \inf_{x \in C_p} |f_0(x)| \geq |f_0(p)|. \tag{4}$$

This is a natural generalization of the **Condition A** in section six of [22]. Actually, the **UCC** requires that the neighborhood of the zero set of f_0 should not be too bad. There are many functions satisfying **UCC**. For example, near each zero, f_0 has the expansion

$$f_0(x) = p_{2l}(x) + O(|x|^{2l+1}),$$

where p_{2l} is the homogeneous polynomial of degree $2l$ and $p_{2l}(x) < 0$ for all $x \neq 0$ and $l \geq 1$. In particular, for $p_{2l}(x) \leq 0$ and $p_{2l}(x) \not\equiv 0$, the remain term $f_0 - p_{2l}$ with certain good behavior implies that **UCC** holds.

Based on our sharp estimate (Theorem 2.4 below), we can improve Struwe's results (Theorem 1.2 and Theorem 6.1 in [22]) as follows:

Theorem 1.2. *Let $f_0 \leq 0$ be a smooth, non-constant function satisfying* **UCC**, *then for suitable $\lambda_k \downarrow 0$, for $u_k = u_{\lambda_k}$ as above and suitable $r_k \downarrow 0$, $p_k \to p_\infty \in M$ with $f_0(p_\infty) = 0$ as $k \to \infty$, the following holds:*

(i) $u_k \to -\infty$ is locally uniformly on compact domains of $M_\infty := M \backslash \{p_\infty\}$.
(ii) We have in local Euclidean coordinates x around $p_k = 0$,

$$\tilde{u}_k(x) := u_k(r_k x) + \log r_k \to \tilde{u}_\infty(x) = \log\left(\frac{4}{4 + 4\pi\eta(p_\infty)|x|^2}\right)$$

in $H^2_{loc}(\mathbb{R}^2)$.

Remark 1.3. Our final goal is to remove the condition **UCC** and we tend to believe that there exists only one bubble for all cases.

Conjecture 1.4. *Let $f_0 \leq 0$ be a smooth, non-constant function, then for suitable $\lambda_k \downarrow 0$, suitable $r_k \downarrow 0$ and $p_k \to p_\infty \in M$ with $f_0(p_\infty) = 0$ as $k \to \infty$, the sequence $u_k = u_{\lambda_k}$ has the following properties:*

(i) $u_k \to -\infty$ is locally uniformly on compact domains of $M_\infty := M \backslash \{p_\infty\}$.

(ii) We have in local Euclidean coordinates x around $p_k = 0$,

$$\tilde{u}_k(x) := u_k(r_k x) + \log r_k \to \tilde{u}_\infty(x) = \log\left(\frac{4}{4 + 4\pi\eta(p_\infty)|x|^2}\right)$$

in $H^2_{loc}(\mathbb{R}^2)$.

The paper is organized as follows. In section 2, we provide some tools stated as some lemmas for later reference. In particular, Theorem 2.4 and Lemma 2.6 should play some crucial roles in our proofs of main theorems. Section 3 will contribute the proofs of our main results.

2. "Bubble" analysis.

Lemma 2.1. *β_λ is strictly decreasing in $(0, -\bar{f}_0)$.*

Proof. Suppose $0 < \lambda_1 < \lambda_2 < -\bar{f}_0$. Due to the well-known result of Kazdan and Warner [17], we know that β_λ is achieved by some $u_\lambda \in C^*_\lambda$. Set $\xi(t) = \int_M f_{\lambda_2} e^{2(1-t)u_{\lambda_1}} \mathrm{d}\mu_{g_b}$. Notice that

$$\xi(0) = \int_M (f_{\lambda_1} + (\lambda_2 - \lambda_1)\eta) e^{2u_{\lambda_1}} \mathrm{d}\mu_{g_b} = \lambda_2 - \lambda_1 > 0$$

and $\xi(1) < 0$. Making use of the continuity of $\xi(t)$, there exists $t_1 \in (0,1)$ such that $\xi(t_1) = 0$. Hence,

$$\beta_{\lambda_2} \leq \frac{1}{2}\int_M |(1-t_1)\nabla u_{\lambda_1}|^2 \mathrm{d}\mu_{g_b} = (1-t_1)^2 \beta_{\lambda_1} < \beta_{\lambda_1}.$$

□

Remark 2.2. This property is first discovered by Galimberti in [12], but his proof is very technical. Later, Struwe [22] gives a simple proof. Our new proof will play an important role in the argument for Theorem 2.4.

Making use of Lebesgue's Theorem, we observe that β_λ is almost everywhere differentiable. Lemma 4.1 in [22] provides an estimate of the derivative via a nice perturbation. Here, we use a different perturbation to improve the estimate.

Lemma 2.3. *β_λ is Lipschitz continuous on $(0, -\bar{f}_0)$ and $\beta'_\lambda = -\frac{\alpha_\lambda}{2}$ a.e.*

Proof. Multiply f_λ to both sides of the equation (2) and integrate it over M to get

$$\alpha_\lambda = \frac{\int_M \langle \nabla u_\lambda, \nabla f_\lambda \rangle \mathrm{d}\mu_{g_b}}{\int_M f_\lambda^2 e^{2u_\lambda} \mathrm{d}\mu_{g_b}} = \frac{\int_M \langle \nabla u_\lambda, \nabla f_0 \rangle \mathrm{d}\mu_{g_b} + \int_M \lambda \langle \nabla u_\lambda, \nabla \eta \rangle \mathrm{d}\mu_{g_b}}{\int_M f_\lambda^2 e^{2u_\lambda} \mathrm{d}\mu_{g_b}}.$$

Using Hölder's inequality, one has

$$0 < \|f_\lambda\|^2_{L^1} \leq \left(\int_M f_\lambda^2 e^{2u_\lambda} \mathrm{d}\mu_{g_b}\right) \int_M e^{-2u_\lambda} \mathrm{d}\mu_{g_b}.$$

Thus,

$$\alpha_\lambda \leq \frac{\|\nabla u_\lambda\|_{L^2}(\|\nabla f_0\|_{L^2} + \lambda \|\nabla \eta\|_{L^2})}{\|f_\lambda\|^2_{L^1}} \int_M e^{-2u_\lambda} \mathrm{d}\mu_{g_b}.$$

With help of Moser-Trudinger's inequality and the fact that $u_\lambda \in C^*_\lambda$, we have

$$\int_M e^{-2u_\lambda} \mathrm{d}\mu_{g_b} = \int_M e^{-2u_\lambda} \mathrm{d}\mu_{g_b} \int_M \eta e^{2u_\lambda} \mathrm{d}\mu_{g_b} \leq C e^{\frac{1}{2\pi}\int_M |\nabla u_\lambda|^2 \mathrm{d}\mu_{g_b}},$$

where C is a positive constant just depending on (M, g_b). Combine these estimates to get

$$\alpha_\lambda \leq C\frac{(\|\nabla f_0\|_{L^2}+\lambda\|\nabla\eta\|_{L^2})}{\|f_\lambda\|_{L^1}^2}\sqrt{2\beta_\lambda}e^{\frac{1}{\pi}\beta_\lambda}.$$

Now, for $\lambda \in [a,b] \subset (0, -\bar{f}_0)$, we construct a function u such that $\int_M f_a e^{2u}\mathrm{d}\mu_{g_b} > 0$ to control the upper bound β_λ, with help from the trick in Lemma 2.1 as

$$\int_M f_\lambda e^{2u}\mathrm{d}\mu_{g_b} \geq \int_M f_a e^{2u}\mathrm{d}\mu_{g_b}.$$

Suppose that f_0 achieves a maximum value at x_0, then there exist $\delta_a > 0$ such that for $x \in B_{\delta_a}(x_0)$, one has $f_a(x) = f_0 + a \geq \frac{a}{2}$. Choose a cut-off function $u_a(x) = s > 0$ to choose later in $B_{\delta_a/2}(x_0)$ and vanish outside $B_{\delta_a}(x_0)$. By letting s be big enough, one has $\int_M f_a e^{2u_a}\mathrm{d}\mu_{g_b} > 0$. Thus, α_λ and β_λ are both uniformly bounded on the compact subset of $(0, -\bar{f}_0)$.

Note that, for δ close to 0 and μ close to λ with $\mu, \lambda \in [a,b]$, one has the following good estimate:

$$\begin{aligned}
\int_M f_\mu e^{2(1+\delta)u_\lambda}\mathrm{d}\mu_{g_b} =& \int_M (f_\lambda + (\mu-\lambda)\eta)e^{2(1+\delta)u_\lambda}\mathrm{d}\mu_{g_b}\\
=& \int_M f_\lambda e^{2u_\lambda}(e^{2\delta u_\lambda}-1)\mathrm{d}\mu_{g_b}\\
&+(\mu-\lambda)\int_M \eta e^{2u_\lambda}(e^{2\delta u_\lambda}-1)\mathrm{d}\mu_{g_b} + \mu - \lambda\\
=&2\delta\int_M f_\lambda e^{2u_\lambda}u_\lambda + O(\delta^2) + (\mu-\lambda)O(\delta) + \mu - \lambda\\
=&\frac{4\beta_\lambda}{\alpha_\lambda}\delta + O(\delta^2) + (\mu-\lambda)O(\delta) + \mu - \lambda,
\end{aligned}$$

where we have used equation (2) and the fact $u_\lambda \in C_\lambda^*$. Hence, if we choose

$$\delta = -\frac{\alpha_\lambda}{4\beta_\lambda}(\mu-\lambda) + O((\mu-\lambda)^2)$$

such that

$$\int_M f_\mu e^{2(1+\delta)u_\lambda}\mathrm{d}\mu_{g_b} = 0,$$

then

$$\beta_\mu \leq (1+\delta)^2\beta_\lambda = \beta_\lambda - \frac{\alpha_\lambda}{2}(\mu-\lambda) + O((\mu-\lambda)^2).$$

In the same manner, we may get

$$\beta_\lambda \leq \beta_\mu - \frac{\alpha_\mu}{2}(\lambda-\mu) + O((\mu-\lambda)^2).$$

These two inequalities imply that

$$-\frac{\alpha_\mu}{2}(\mu-\lambda) + O((\mu-\lambda)^2) \leq \beta_\mu - \beta_\lambda \leq -\frac{\alpha_\lambda}{2}(\mu-\lambda) + O((\mu-\lambda)^2). \tag{5}$$

The estimate (5) can be used to show that β_λ is Lipschitz continuous. In fact, more can be done:

$$\frac{\mathrm{d}\beta_\lambda}{\mathrm{d}\lambda}(\lambda_+) \leq -\frac{\alpha_\lambda}{2}$$

and

$$\frac{\mathrm{d}\beta_\lambda}{\mathrm{d}\lambda}(\lambda_-) \geq -\frac{\alpha_\lambda}{2}.$$

Since β_λ is almost everywhere differentiable, one has

$$\beta'_\lambda = -\frac{\alpha_\lambda}{2} \quad a.e.$$

□

Now, we turn to the crucial estimate in this paper.

Theorem 2.4.

$$\limsup_{\lambda \downarrow 0} \frac{\beta_\lambda}{\log(1/\lambda)} \leq 2\pi.$$

Proof. Let $p_0 \in M$ be the maximum point for $f_0(p_0) = 0$ and $\eta(p_0) > 0$. Let $\alpha > 0$ be fixed. We assume that we have local normal coordinates near $p_0 = 0$. There exists $C_1 > 0$ depending on f_0 such that, for a suitable constant $L > 0$ to be chosen later, on the disk $B_{\lambda^\alpha/L}(0)$, f_0 has the form

$$f_0(x) = \frac{1}{2} Hess_{f_0}|_{p_0}(x,x) + O(|x|^3) \geq -C_1|x|^2$$

and the function η satisfies the estimate

$$\eta(x) \geq \frac{1}{2}\eta(p_0) > 0.$$

Consider the function $z_\lambda(Lx/\lambda^\alpha)$ defined by $z_\lambda(x) = \log(1/|x|)$ for $\lambda^{1/2} \leq |x| \leq 1$ and $z_\lambda(x) = \frac{1}{2}\log(1/\lambda)$ for $|x| \leq \lambda^{1/2}$. Set $w_\lambda(x) = z_\lambda(Lx/\lambda^\alpha)$. Since $z_\lambda \in H_0^1(B_1(0))$, we conclude that

$$\|\nabla w_\lambda\|_{L^2}^2 = \|\nabla z_\lambda\|_{L^2}^2 = \pi \log(1/\lambda). \tag{6}$$

Extending $w_\lambda(x)$ to be zero outside $B_{\lambda^\alpha/L}(0)$ for sufficiently small $\lambda > 0$, one has

$$\begin{aligned}
\int_M f_\lambda e^{2(2+2\alpha)w_\lambda} \mathrm{d}\mu_{g_b} &= \int_M f_\lambda \mathrm{d}\mu_{g_b} + \int_{B_{\lambda^\alpha/L}(0)} f_\lambda (e^{2(2+2\alpha)w_\lambda} - 1) \mathrm{d}\mu_{g_b} \\
&= \int_M f_0 \mathrm{d}\mu_{g_b} + \lambda + \int_{B_{\lambda^\alpha/L}(0)} \lambda \eta e^{2(2+2\alpha)w_\lambda} \mathrm{d}\mu_{g_b} \\
&\quad + \int_{B_{\lambda^\alpha/L}(0)} f_0 e^{2(2+2\alpha)w_\lambda} \mathrm{d}\mu_{g_b} - \int_{B_{\lambda^\alpha/L}(0)} f_\lambda \mathrm{d}\mu_{g_b} \\
&\geq \int_M f_0 \mathrm{d}\mu_{g_b} + \lambda + \lambda \frac{1}{2}\eta(p_0) \int_{B_{\lambda^\alpha/L}(0)} e^{2(2+2\alpha)w_\lambda} \mathrm{d}\mu_{g_b} \\
&\quad - C_1 \int_{B_{\lambda^\alpha/L}(0)} |x|^2 e^{2(2+2\alpha)w_\lambda} \mathrm{d}\mu_{g_b} - \lambda(\max_M \eta)\pi \frac{\lambda^{2\alpha}}{L^2}.
\end{aligned}$$

Now, we will deal with the terms of the right side one by one. First of all, let $y = Lx/\lambda^\alpha$ to get

$$\begin{aligned}
\lambda \int_{B_{\lambda^\alpha/L}(0)} e^{2(2+2\alpha)w_\lambda} \mathrm{d}\mu_{g_b} &= \frac{\lambda^{1+2\alpha}}{L^2} \int_{B_1(0)} e^{(4+4\alpha)z_\lambda} \mathrm{d}y \\
&= \frac{\lambda^{1+2\alpha}}{L^2} \left(\pi \lambda^{-1-2\alpha} + 2\pi \int_{\lambda^{1/2}}^1 r^{-3-4\alpha} \mathrm{d}r \right) \\
&= \frac{\pi}{L^2} + \frac{\pi}{L^2(1+2\alpha)} - \frac{\pi \lambda^{1+2\alpha}}{L^2(1+2\alpha)}.
\end{aligned}$$

Second, one also has

$$\begin{aligned}\int_{B_{\lambda^\alpha/L}(0)}|x|^2e^{2(2+2\alpha)w_\lambda}\mathrm{d}\mu_{g_b}&=\frac{\lambda^{4\alpha}}{L^4}\int_{B_1(0)}|y|^2e^{(4+4\alpha)z_\lambda}\mathrm{d}y\\&=\frac{\lambda^{4\alpha}}{L^4}\left(\lambda^{-2-2\alpha}2\pi\int_0^{\lambda^{1/2}}r^3\mathrm{d}r+2\pi\int_{\lambda^{1/2}}^1r^{-1-4\alpha}\mathrm{d}r\right)\\&=\frac{\pi}{2L^4}(1+1/\alpha)\lambda^{2\alpha}-\frac{\pi}{2\alpha L^4}\lambda^{4\alpha}.\end{aligned}$$

Now, choose $L=\sqrt{\frac{\pi\eta(p_0)}{-4\int_M f_0\mathrm{d}\mu_{g_b}}}$ to get

$$\int_M f_\lambda e^{2(2+2\alpha)w_\lambda}\mathrm{d}\mu_{g_b}\geq-\int_M f_0\mathrm{d}\mu_{g_b}-C(f_0,\alpha)(\lambda^{2\alpha}+\lambda^{1+2\alpha}),$$

where $C(f_0,\alpha)>0$ is a constant depending on f_0 and α. Hence, there exists small $\lambda_0>0$ such that, for $\lambda<\lambda_0$, one has

$$\int_M f_\lambda e^{2(2+2\alpha)w_\lambda}\mathrm{d}\mu_{g_b}>0.$$

Now, let us consider the real valued function $\varphi(t)=\int_M f_\lambda e^{2(2+2\alpha)(1-t)w_\lambda}\mathrm{d}\mu_{g_b}$ on the interval $\in[0,1]$. Notice that $\varphi(0)>0$ and $\varphi(1)<0$. By the intermediate value theorem for continuous functions, there exists $t_0\in(0,1)$ such that $\varphi(t_0)=0$. Hence, with the help of (6), one has

$$\beta_\lambda\leq\frac{1}{2}\|(1-t_0)(2+2\alpha)\nabla w_\lambda\|^2_{L^2}=2(1+\alpha)^2(1-t_0)^2\pi\log 1/\lambda\leq 2(1+\alpha)^2\pi\log 1/\lambda.$$

Since α is arbitrary, one has

$$\limsup_{\lambda\downarrow 0}\frac{\beta_\lambda}{\log(1/\lambda)}\leq 2\pi.$$

□

Combine Theorem 2.4 and Lemma 2.3 to make the following claim:

$$\liminf_{\lambda\downarrow 0}\lambda\alpha_\lambda\leq 2\liminf_{\lambda\downarrow 0}\lambda|\frac{\mathrm{d}\beta_\lambda}{\mathrm{d}\lambda}|\leq 4\pi. \tag{7}$$

Indeed, suppose the claim is not true, and there exists $\lambda_1>0$ and some constant $c_0>2\pi$ such that for almost $0<\lambda<\lambda_1$, $|\beta'_\lambda|\geq\frac{c_0}{\lambda}$, then for any sufficiently small λ, we have

$$\beta_\lambda-\beta_{\lambda_1}\geq\int_\lambda^{\lambda_1}|\beta'_s|\mathrm{d}s\geq c_0\int_\lambda^{\lambda_1}\frac{\mathrm{d}s}{s}\geq c_0\log(1/\lambda)+C>(\pi+c_0/2)\log(1/\lambda)$$

which contradicts Theorem 2.4. Now, we consider the conformal metric $\hat{g}_\lambda=e^{2(u_\lambda+1/2\log\alpha_\lambda)}g_b$ and denote $K_{\hat{g}_\lambda}$ as Gaussian curvature. From equation (2), we have $K_{\hat{g}_\lambda}=f_\lambda$. Moreover, since η is positive and $f_0\leq 0$, it is not hard to prove that

$$|K_{\hat{g}_\lambda}|=|f_\lambda|\leq\lambda|\eta|+|f_0|=\lambda\eta-f_0=2\lambda\eta-f_\lambda.$$

Thus,

$$\liminf_{\lambda\downarrow 0}\int_M|K_{\hat{g}_\lambda}|\mathrm{d}\mu_{\hat{g}_\lambda}\leq 2\liminf_{\lambda\downarrow 0}\lambda\alpha_\lambda\leq 8\pi \tag{8}$$

and

$$\liminf_{\lambda\downarrow 0}\int_M K^+_{\hat{g}_\lambda}\mathrm{d}\mu_{\hat{g}_\lambda}\leq\liminf_{\lambda\downarrow 0}\lambda\alpha_\lambda\leq 4\pi. \tag{9}$$

Now, we choose a subsequence $\lambda_k \to 0$ such that

$$\lim_{k\to\infty} \lambda_k \alpha_{\lambda_k} \leq 4\pi. \tag{10}$$

For simplicity, set $u_k = u_{\lambda_k}$ and $g_k = e^{2u_k} g_b$. Thus, equation (2) takes the form

$$-\Delta_{g_b} u_k = \alpha_{\lambda_k} f_{\lambda_k} e^{2u_k}.$$

Lemma 2.5. *For this sequence $\{u_k\}$, we have the following properties.*

(i) There holds $\alpha_{\lambda_k} \to \infty$ as $k \to \infty$.

(ii) There exists finitely many points p_i with $1 \leq i \leq i_0$ and $f_0(p_i) = 0$, such that for any $s > 0$,

$$\liminf_{k\to\infty} \int_{B_s(p_i)} K^+_{g_k} e^{2u_k} \mathrm{d}\mu_{g_b} \geq \pi. \tag{11}$$

(iii) There holds

$$\lim_{k\to\infty} \bar{u}_k = -\infty$$

and for any $B_r(x) \subset\subset M \backslash \{p_i; 1 \leq i \leq i_0\}$,

$$\|(u_k - \bar{u}_k)^+\|_{L^\infty(B_r(x))} \leq C.$$

Proof. Based on the fact that $u_k \in C^*_{\lambda_k}$, there holds

$$\int_M (-f_0) e^{2u_k} \mathrm{d}\mu_{g_b} = \lambda_k \int_M \eta e^{2u_k} \mathrm{d}\mu_{g_b} = \lambda_k.$$

Apply Jensen's inequality to get

$$\exp\left(\int_M 2u_k \frac{(-f_0)}{\|f_0\|_{L^1(M)}} \mathrm{d}\mu_{g_b}\right) \leq \int_M e^{nu_k} \frac{-f_0}{\|f_0\|_{L^1(M)}} \mathrm{d}\mu_{g_b} = \frac{\lambda_k}{\|f_0\|_{L^1(M)}}.$$

Thus, one has

$$\int_M (-f_0) u_k \mathrm{d}\mu_{g_b} \to -\infty, \quad \text{as} \quad k \to \infty.$$

Notice the facts that

$$\begin{aligned} |\int_M (-f_0) u_k^- \mathrm{d}\mu_{g_b} + \int_M (-f_0) u_k \mathrm{d}\mu_{g_b}| &= \int_M (-f_0) u_k^+ \mathrm{d}\mu_{g_b} \\ &\leq \frac{\max_M(-f_0)}{2} \int_M e^{2u_k} \mathrm{d}\mu_{g_b} \leq C, \end{aligned}$$

and

$$\int_M (-f_0) u_k^- \mathrm{d}\mu_{g_b} \leq \max_M(-f_0) \int_M u_k^- \mathrm{d}\mu_{g_b}.$$

These facts easily yield the conclusion

$$\int_M u_k^- \mathrm{d}\mu_g \to +\infty, \quad \text{as} \quad k \to \infty.$$

By making use of the simple fact that $x \leq e^x$, one conclude that

$$\begin{aligned} \int_M u_k \mathrm{d}\mu_{g_b} &= \int_M u_k^+ \mathrm{d}\mu_{g_b} - \int_M u_k^- \mathrm{d}\mu_{g_b} \\ &\leq \frac{1}{2} \int_M e^{2u_k} \mathrm{d}\mu_{g_b} - \int_M u_k^- \mathrm{d}\mu_{g_b} \\ &\leq \frac{1}{2\min_M \eta} - \int_M u_k^- \mathrm{d}\mu_{g_b} \to -\infty. \end{aligned}$$

In other words, one has

$$\bar{u}_k \to -\infty. \tag{12}$$

Let us recall a well known fact. On Riemann surface (T^2, g_h), there exists a positive Green's function satisfying

$$-\Delta_{g_b} G(x,y) = \delta_y - 1$$

with the estimate

$$|G(x,y) - \frac{1}{2\pi}\log\frac{1}{|x-y|}| \le C$$

near the diagonal of $M \times M$. Details can be found in Chapter 4 of the reference [1]. From now on, we should adopt the convention that a constant C may be different from line to line. Suppose for a point $x_0 \in M$, there exists $r_0 > 0$ and small $a_0 > 0$ such that

$$\int_{B_{3r_0}(x_0)} K_{g_k}^+ e^{2u_k} \mathrm{d}\mu_g \le \pi - a_0 \tag{13}$$

for sufficiently large k. Thus, for any $x \in B_{r_0}(x_0)$, by Green's representation, we have

$$\begin{aligned}
&u_k(x) - \bar{u}_k \\
=& \int_M G(x,y) K_{g_k} e^{2u_k(y)} \mathrm{d}\mu_{g_b} \\
\le& \int_M G(x,y) K_{g_k}^+ e^{2u_k(y)} \mathrm{d}\mu_{g_b} \\
=& \int_{B_{2r_0}(x_0)} G(x,y) K_{g_k}^+(y) e^{2u_k(y)} \mathrm{d}\mu_{g_b}(y) + \int_{M\setminus B_{2r_0}(x_0)} G(x,y) K_{g_k}^+(y) e^{2u_k(y)} \mathrm{d}\mu_{g_b}(y) \\
\le& \int_{B_{2r_0}(x_0)} G(x,y) K_{g_k}^+ e^{2u_k(y)} \mathrm{d}\mu_{g_b}(y) + C,
\end{aligned}$$

where we have used the fact $|G(x,y)| \le C$ on $B_{r_0}(x_0) \times (M \setminus B_{2r_0}(x_0))$ and $\|K_{g_k}^+ e^{2u_k}\|_{L^1} \le C$ (see the equation (9)). Set $h_k = K_{g_k}^+ e^{2u_k}$ for brevity. Choose a cut-off function $0 \le \xi \le 1$ with $\xi \equiv 1$ in $B_{2r_0}(x_0)$ and ξ vanishes outside $B_{3r_0}(x_0)$. For $\alpha > 0$, take the same strategy as that in [6] and use Jensen's inequality to get

$$\begin{aligned}
&\int_{B_{r_0}(x_0)} e^{\alpha(u_k(x) - \bar{u}_k)} \mathrm{d}\mu_{g_b} \\
\le& C \int_{B_{r_0}(x_0)} \exp\left(\int_{B_{2r_0}(x_0)} \alpha G(x,y) h_k(y) \mathrm{d}\mu_{g_b}(y)\right) \mathrm{d}\mu_{g_b}(x) \\
\le& C \int_{B_{r_0}(x_0)} \exp\left(\int_M \alpha |G(x,y)| \cdot |h_k(y)\xi(y)| \mathrm{d}\mu_{g_b}(y)\right) \mathrm{d}\mu_{g_b}(x) \\
\le& C \int_{B_{r_0}(x_0)} \int_M \exp\left(\alpha \|h_k\xi\|_{L^1(M)} |G(x,y)|\right) \frac{|h_k\xi|}{\|h_k\xi\|_{L^1(M)}} \mathrm{d}\mu_{g_b}(y) \mathrm{d}\mu_{g_b}(x) \\
\le& C \int_M \frac{|h_k\xi|}{\|h_k\xi\|_{L^1(M)}} \int_M \left(\frac{1}{|x-y|}\right)^{\frac{\alpha\|h_k\xi\|_{L^1(M)}}{2\pi}} \mathrm{d}\mu_{g_b}(x) \mathrm{d}\mu_{g_b}(y).
\end{aligned}$$

With the help of (13), there exists $\alpha_0 > 2$ such that the last integral converges. Therefore, for any $x \in B_{r_0/2}(x_0)$, with the help of (10), one has

$$u_k(x) - \bar{u}_k$$

$$
\begin{aligned}
&\le C + C\int_{B_{r_0}(x_0)} G(x,y)e^{2u_k}\mathrm{d}\mu_{g_b}(y)\\
&\le C + C\left(\int_{B_{r_0}(x_0)} e^{\alpha_0 u_k}\mathrm{d}\mu_{g_b}\right)^{2/\alpha_0}\left(\int_{B_{r_0}(x_0)} |G|^{\frac{1}{1-2/\alpha_0}}\mathrm{d}\mu_{g_b}\right)^{1-2/\alpha_0}\\
&\le C,
\end{aligned}
$$

which implies that

$$
\|(u_k-\bar{u}_k)^+\|_{L^\infty(B_{r_0/2}(x_0))} \le C. \tag{14}
$$

If for all $x \in M$ the estimate (13) holds, since M is compact, there exists finitely many balls covering M and one can conclude that one has

$$
\|(u_k-\bar{u}_k)^+\|_{L^\infty(M)} \le C. \tag{15}
$$

However, this is not possible. Supposing that (15) holds true, we first claim that $\{\bar{u}_k\}$ is bounded. Observe that

$$
\int_M \eta e^{2u_k}\mathrm{d}\mu_{g_b} = 1.
$$

This implies the upper bound of $\bar{u}_k$ by Jensen's inequality

$$
\bar{u}_k \le \frac{1}{2}\log\int_M e^{2u_k}\mathrm{d}\mu_{g_b} \le \frac{1}{2}\log\frac{1}{\min_M \eta}.
$$

On the other hand, the estimate (14) yields

$$
\frac{1}{\max_M \eta} \le \int_M e^{2u_k}\mathrm{d}\mu_{g_b} = \int_M e^{2(u_k-\bar{u}_k)}\mathrm{d}\mu_{g_b} e^{2\bar{u}_k} \le Ce^{2\bar{u}_k},
$$

which implies $\bar{u}_k \ge -C$ for some constant C. Combine those two estimates to get a desired contradiction with (12).

With help of (9), there are only finitely many points $p_i \in M$ with $1 \le i \le J$, $J \in \mathbb{N}$ such that (13) does not hold. Since

$$
\int_{B_s(p_i)} K_{g_k}^+ e^{2u_k}\mathrm{d}\mu_{g_b} \le \alpha_{\lambda_k}\lambda_k\int_M e^{2u_k}\mathrm{d}\mu_{g_b} \le \frac{1}{\min_M \eta}\alpha_{\lambda_k}\lambda_k,
$$

we must have $\alpha_{\lambda_k} \to \infty$ as $k \to \infty$.

Next, we claim that $f(p_i) = 0$ for each $1 \le p_i \le J$. Suppose for some $1 \le i_0 \le J$ $f_0(p_{i_0}) < 0$, there exists $s > 0$ such that for any $x \in B_s(p_{i_0})$ and sufficient large k,

$$
K_{g_k} = \alpha_{\lambda_k}(\lambda_k + f_0(x)) \le 0,
$$

then

$$
\int_{B_s(p_{i_0})} K_{g_k}^+ e^{2u_k}\mathrm{d}\mu_{g_b} \le 0,
$$

which contradicts (11).

From the above argument, for any $x \in M\backslash\{p_i; 1 \le i \le J\}$, there exists $r > 0$ such that

$$
B_r(x) \subset\subset M\backslash\{p_i; 1 \le i \le J\}
$$

and

$$
\|(u_k-\bar{u}_k)^+\|_{L^\infty(B_r(x))} \le C.
$$

This completes the proof of our lemma. $\square$

In particular, by Lemma 2.5 and the estimate (10), we obtain the estimate

$$\pi - o(1) \leq \alpha_{\lambda_k}\lambda_k \leq 4\pi + o(1) \tag{16}$$

for sufficiently large k.

Now, we establish a modified Ding's lemma (See Lemma 1.1, [7]), which will be used later. It has also been established in [15](Proposition 8.5).

Lemma 2.6. *If u is a weak solution to the equation*

$$-\Delta u = (1+f)e^u \quad \text{in } \mathbb{R}^2$$

where $f \leq 0$, $f \in L^\infty_{loc}(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2} |1+f|e^u \mathrm{d}x < +\infty$, then

$$\int_{\mathbb{R}^2} e^u \mathrm{d}x \geq 8\pi,$$

with " $=$ " iff $f = 0$ a.e..

Proof. First, if $\int_{\mathbb{R}^2} e^u \mathrm{d}x = +\infty$, nothing needs to be done. Now, we consider the conformal metric with finite volume i.e.

$$\int_{\mathbb{R}^2} e^u \mathrm{d}x < +\infty. \tag{17}$$

Define upper level set $\Omega_t = \{x|u(x) > t\}$ for any t. Hence, with the measure $e^u dx$, $|\Omega_t| < +\infty$ due to (17). Based on the results of [6] and our assumptions, it is not hard to see that for any $p > 2$, we have $u \in W^{2,p}_{loc}(\mathbb{R}^2)$. Following the argument of Lemma 1.1 in [7] and with help of the co-area formula as well as Sard's lemma, we have

$$\int_{\Omega_t} e^u \mathrm{d}x \geq \int_{\Omega_t} (1+f)e^u \mathrm{d}x = -\int_{\Omega_t} \Delta u \mathrm{d}x = \int_{\partial\Omega_t} |\nabla u| \mathrm{d}s \tag{18}$$

and

$$-\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t| = \int_{\partial\Omega_t} \frac{\mathrm{d}s}{|\nabla u|}.$$

By the Schwarz inequality and the isoperimetric inequality,

$$\int_{\partial\Omega_t} \frac{\mathrm{d}s}{|\nabla u|} \int_{\partial\Omega_t} |\nabla u| \mathrm{d}s \geq |\partial\Omega_t|^2 \geq 4\pi|\Omega_t|.$$

Hence,

$$-\frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t| \int_{\Omega_t} e^u \mathrm{d}x \geq 4\pi|\Omega_t|$$

and so,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\Omega_t} e^u \mathrm{d}x\right)^2 = 2e^t \frac{\mathrm{d}}{\mathrm{d}t}|\Omega_t| \int_{\Omega_t} e^u \mathrm{d}x \leq -8\pi e^t |\Omega_t|.$$

Integrate from $-\infty$ to $+\infty$ to get

$$-\left(\int_{\mathbb{R}^2} e^u \mathrm{d}x\right)^2 \leq -8\pi \int_{\mathbb{R}^2} e^u \mathrm{d}x,$$

which yields

$$\int_{\mathbb{R}^2} e^u \mathrm{d}x \geq 8\pi.$$

If the equal sign holds, then the equal sign in (18) also holds a.e., which implies $f = 0$ a.e. Conversely, if $f = 0$ a.e., by the classification theorem in [7], we have $\int_{\mathbb{R}^2} e^u \mathrm{d}x = 8\pi$. □

Remark 2.7. Actually, it is a global version of Alexandrov-Bol's inequality (see [2]). Recently, there are a lot of works devoted to its study including [23], [14], [16] and many others. If we do not care about the rigidity of this estimate, we could deduce lower bounds of conformal volume from Alexandrov-Bol's inequality

$$(\int_{\partial B_r(0)} e^{u/2}\mathrm{d}s)^2 \geq \frac{1}{2}(8\pi - \int_{B_r(0)} e^u\mathrm{d}x)\int_{B_r(0)} e^u\mathrm{d}x, \tag{19}$$

provided the differential inequality holds: $-\Delta u \leq e^u$. Usually, we assume that $u \in C^2$, but for less regularity such as $u \in W^{2,p}$ with $p > 2$, the inequality still holds (see [3], [4]), even with some singularities.

To see this, we argue by contradiction. Suppose that $\int_{\mathbb{R}^2} e^u\mathrm{d}x < 8\pi$, and Alexandrov-Bol's inequality (19) shows that

$$(\int_{\partial B_r(0)} e^{u/2}\mathrm{d}s)^2 \geq c_1\int_{B_r(0)} e^u\mathrm{d}x,$$

where $c_1 = \frac{1}{2}(8\pi - \int_{\mathbb{R}^2} e^u\mathrm{d}x) > 0$. Based on Hölder's equality, we have

$$c_1\int_{B_r(0)} e^u\mathrm{d}x \leq (\int_{\partial B_r(0)} e^{u/2}\mathrm{d}s)^2 \leq 2\pi r\int_{\partial B_r(0)} e^u\mathrm{d}s = 2\pi r\frac{\partial}{\partial r}\int_{B_r(0)} e^u\mathrm{d}x,$$

then

$$(\log\int_{B_r(0)} e^u\mathrm{d}x)' \geq \frac{c_1}{2\pi r},$$

which will yield $\int_{\mathbb{R}^2} e^u\mathrm{d}x = +\infty$ by integrating it over $[1, +\infty)$. Thus, we get the contradiction.

3. Proof of main theorems.

With above preparations, we now return to prove our main claims.

First of all, as $k \to \infty$, we have

$$K_{g_k}^+ e^{u_k}\mathrm{d}\mu_{g_b} \rightharpoonup K_+ + \sum_{i\in I}\gamma_i\delta_{p_i}$$

weakly-$*$ in the sense of the measures, with a measure $K_+ \geq 0$ on (M, g_b) having no atoms and with, at most, countably many atoms of weight $\gamma_i > 0, i \in I \subset \mathbb{N}$. Note that (9) gives the bound $\sum_i \gamma_i \leq 4\pi$. By Lemma 2.5, we have $\gamma_i \geq \pi$ and $1 \leq i \leq i_0$.

Given any set $\Omega \subset\subset M_\infty := M\backslash\{p_i; 1 \leq i \leq i_0\}$, there exists finitely many balls $B_r(x) \subset\subset M_\infty$ that cover Ω, then Lemma 2.5 implies that

$$\|(u_k - \bar{u}_k)^+\|_{L^\infty(\Omega)} \leq C$$

and

$$u_k \to -\infty$$

uniformly on Ω. Thus, the first part of our theorems 1.1 and 1.2 hold.

Next, we focus on the second part of both theorems. Although Theorem 1.2 is a more general case, we could get the estimate of the scale r_k compared with λ_k when f_0 satisfies the non-degenerate condition. We take different strategies to do blow-up analysis.

Proof of Theorem 1.1:

Proof. Let point p_0 be one of the concentration points $p_\infty^{(i)}, 1 \leq i \leq i_0$. There exists $R > 0$ and $p_k \to p_0$ such that u_k, for sufficiently large $k \in \mathbb{N}$, attains its maximum on $B_R(p_0)$ at p_k. In local normal coordinates x near p_0, $p_k = x_k$ and $\Delta u_k(x_k) \leq 0$. From

$$-\Delta u_k = \alpha_{\lambda_k} f_{\lambda_k} e^{2u_k},$$

it follows that $\alpha_{\lambda_k} f_{\lambda_k}(x_k) \geq 0$; that is $-f_0(x_k) \leq \lambda_k \eta(x_k)$. Note that $f_0(p_0)$ is of non-degenerate of order $2l$ and there exists $C_1 > 0$ such that $-f_0(x) > \lambda_k \max_M \eta$ for $|x|^{2l} \geq C_1 \lambda_k, |x| \leq R$. It follows that $|x_k|^{2l} \leq C_1 \lambda_k$. Lemma 2.5 shows that, for sufficiently large k, there holds

$$\pi/2 \leq \int_{B_R(0)} K_{g_k}^+ \mathrm{d}\mu_{g_k} = \int_{B_{(C_1\lambda_k)^{\frac{1}{2l}}}(0)} \alpha_{\lambda_k} f_{\lambda_k}^+ e^{2u_k} \mathrm{d}\mu_{g_b} \leq C_1^{1/l} \pi \alpha_{\lambda_k} \lambda_k^{1+1/l} e^{2u_k(x_k)}. \tag{20}$$

By a Liouville-type theorem, Struwe ruled out the case $\alpha_{\lambda_k} \lambda_k^{1+1/l} e^{2u_k(x_k)} \leq C$ with $l = 1$; see Theorem 4.5 in [22]. However, we have to use a different method, mainly with help from Lemma 2.6. Suppose

$$C^{-1} \leq \alpha_{\lambda_k} \lambda_k^{1+1/l} e^{2u_k(x_k)} \leq C$$

for all $k \in \mathbb{N}$. In view of (16), we have $|2u_k(x_k) + \frac{1}{l} \log \lambda_k| \leq C$. Set $r_k = \lambda_k^{\frac{1}{2l}}$ and

$$\tilde{u}_k(x) = u_k(r_k x) + \log r_k,$$

then

$$\tilde{u}_k(x) \leq \tilde{u}_k(x_k/r_k) = u_k(x_k) + \frac{1}{2l} \log \lambda_k \leq C$$

on $D_k = \{x||r_k x| < R\}$. Moreover, $\tilde{u}_k$ satisfies the equation

$$-\Delta \tilde{u}_k = \tilde{f}_k e^{2\tilde{u}_k}$$

with

$$\tilde{f}_k = \alpha_{\lambda_k} \lambda_k (\eta(r_k x) + \lambda_k^{-1} f_0(r_k x)).$$

By the assumption on f_0, we have up to a subsequence on each compact subset

$$\tilde{f}_k = \alpha_{\lambda_k} \lambda_k (\eta(r_k x) + p_{2l}(x) + r_k O(|x|^{2l+1})) \to \mu(\eta(p_0) + p_{2l}(x))$$

with $\pi \leq \mu \leq 4\pi$ due to (16). From Theorem 3 in [6], it follows that, up to a subsequence, $\tilde{u}_k \to \tilde{u}_\infty$ in $H^2_{loc}(\mathbb{R}^2)$, where $\tilde{u}_\infty$ satisfies the equation

$$-\Delta \tilde{u}_\infty = \mu(\eta(p_0) + p_{2l}(x)) e^{2\tilde{u}_\infty}. \tag{21}$$

On the other hand, Fatou's lemma yields that, for any $L \in \mathbb{N}$,

$$\int_{B_L(0)} \eta(p_0) e^{2\tilde{u}_\infty} \mathrm{d}x \leq \liminf_{k\to\infty} \int_{B_L(0)} \eta(r_k x) e^{2\tilde{u}_k} \mathrm{d}x \leq \liminf_{k\to\infty} \int_M \eta e^{2u_k} \mathrm{d}\mu_{g_b} = 1.$$

Thus, letting $L \to \infty$, we find that $e^{2\tilde{u}_\infty} \in L^1(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}^2} \eta(p_0) e^{2\tilde{u}_\infty} \mathrm{d}x = \lim_{L\to\infty} \int_{B_L(0)} \eta(p_0) e^{2\tilde{u}_\infty} \mathrm{d}x \leq 1. \tag{22}$$

Similarly, by (8), we also conclude that $\int_{\mathbb{R}^2} |\mu(\eta(p_0) + p_{2l}(x))| e^{2\tilde{u}_\infty} \mathrm{d}x \leq C$. Finally, we apply Lemma 2.6 to get

$$\int_{\mathbb{R}^2} \mu \eta(p_0) e^{2\tilde{u}_\infty} \mathrm{d}x \geq 4\pi.$$

Combining the above estimates, we must have $\mu \int_{\mathbb{R}^2} \eta(p_0) e^{2\tilde{u}_\infty} \mathrm{d}x = 4\pi$, then Lemma 2.6 shows that $p_{2l} \equiv 0$, which is a desired contradiction.

Thus, $\alpha_{\lambda_k}\lambda_k^{1+1/l}e^{2u_k(x_k)} \to \infty$. Choose r_k such that $r_k^2\alpha_{\lambda_k}\lambda_k e^{2u_k(x_k)} = 1$. It follows that $r_k^{2l}/\lambda_k \to 0$ as $k \to \infty$. Use these r_k to rescale u_k

$$\tilde{u}_k = u_k(x_k + r_k x) - u_k(x_k) \le \tilde{u}_k(0) = 0$$

on $D_k = \{x; |x_k + r_k x| < R\}$. Note that, by the estimate (16), we have $|u_k(x_k) + \log r_k| \le C$ and $\{D_k\}$ exhausts $\mathbb{R}^2$. Thus, $\tilde{u}_k$ satisfies

$$-\Delta\tilde{u}_k = \tilde{f}_k e^{2\tilde{u}_k}, \text{ on } D_k,$$

where

$$\tilde{f}_k(x) = r_k^2\alpha_{\lambda_k} f_{\lambda_k}(x_k + r_k x)e^{2u_k(x_k)} = \eta(x_k + r_k x) + \lambda_k^{-1} f_0(x_k + r_k x)$$

satisfies $\tilde{f}_k(0) \ge 0$. Since $|x_k|^{2l} \le C_1\lambda_k$ and $r_k^{2l}/\lambda_k \to 0$, for a subsequence we have $\lambda_k^{-1} f_0(x_k + r_k x) \to c_0$ for some constant $-1 \le c_0 \le 0$ and $\tilde{f}_k \to \eta(p_0) + c_0 = r_0^2 \ge 0$ locally uniformly on $\mathbb{R}^2$.

In view of the volume bound

$$\int_{D_k} e^{2\tilde{u}_k}\mathrm{d}x = \alpha_{\lambda_k}\lambda_k \int_{B_R(0)} e^{2u_k}\mathrm{d}x \le C, \tag{23}$$

apply Theorem 3 in [6] to conclude that for a subsequence, $\tilde{u}_k \to \tilde{u}_\infty$ locally uniformly, where $\tilde{u}_\infty \le 0 = \tilde{u}_\infty(0)$ solves the equation

$$-\Delta\tilde{u}_\infty = \mu r_0^2 e^{2\tilde{u}_\infty}, \text{ on } \mathbb{R}^2 \tag{24}$$

with $\int_{\mathbb{R}^2} e^{2\tilde{u}_\infty}\mathrm{d}x < \infty$. This implies that $r_0 > 0$. By the classification theorem in [7], we have

$$\tilde{u}_\infty(x) = \log\left(\frac{4}{4 + \mu r_0^2|x|^2}\right).$$

By definition of r_0, at each blow up point $p_\infty^{(i)}$ with $0 < r_0 \le 1$, we have

$$\begin{aligned} 4\pi = \mu r_0^2 \lim_{L\to\infty}\int_{B_L(0)} e^{2\tilde{u}_\infty}\mathrm{d}x &\le \mu\eta(p_0)\lim_{L\to\infty}\int_{B_L(0)} e^{2\tilde{u}_\infty}dx \\ &\le \lim_{k\to\infty}\inf \lambda_k\alpha_{\lambda_k}\int_M \eta e^{2u_k}\mathrm{d}\mu_{g_b} \\ &\le \mu \le 4\pi. \end{aligned}$$

Thus, in each step, the equal sign must hold. In particular,

(a)
$$r_0^2 = \eta(p_0),$$

(b)
$$\lim_{k\to\infty}\lambda_k\alpha_{\lambda_k} = 4\pi, \tag{25}$$

(c)
$$\lambda_k^{-1} f_0(x_k + r_k x) \to 0, \tag{26}$$

locally uniformly on $\mathbb{R}^2$.

Thus, our claim follows. □

Now, we treat the general case. For brevity, we still use the same notations as that in the proof of Theorem 1.1.

Proof of Theorem 1.2:

Proof. Given a point $p_0 = p_i, 1 \le i \le i_0$, in normal coordinates x near $p_0 = 0$, consider a small ball $B_R(0)$ with $R > 0$ sufficiently small. By Lemma 2.5, we can find radii $r_k \to 0$ as $k \to \infty$ such that

$$\int_{B_{r_k}(x_k)} K_{g_k}^+ \mathrm{d}\mu_{g_k} = \sup_{x \in B_R(0)} \int_{B_{r_k}(x)} K_{g_k}^+ \mathrm{d}\mu_{g_k} = \frac{\pi}{3}, \tag{27}$$

Blow it up at x_k to have

$$\tilde{u}_k = u_k(x_k + r_k x) + \log r_k$$

on $D_k = \{x||x_k + r_k x| < R\}$. The domains $\{D_k\}$ exhausts $\mathbb{R}^2$. Thus, the new function $\tilde{u}_k$ satisfies the equation

$$-\Delta \tilde{u}_k = \tilde{f}_k e^{2\tilde{u}_k}, \text{ on } D_k \tag{28}$$

with

$$\tilde{f}_k(x) = \alpha_{\lambda_k} f_{\lambda_k}(x_k + r_k x) = \alpha_{\lambda_k} \lambda_k (\eta(x_k + r_k x) + \lambda_k^{-1} f_0(x_k + r_k x)).$$

Decompose $\tilde{u}_k$ into three parts as $\tilde{u}_k = \tilde{u}_k^{(+)} + \tilde{u}_k^{(0)} + \tilde{u}_k^{(-)}$ on any unit ball $B := B_1(x_0)$, where $x_0 \in D_k$ such that $\Delta \tilde{u}_k^{(0)} = 0$ in B with $\tilde{u}_k^{(0)} = \tilde{u}_k$ on ∂B and $\tilde{u}_k^{(\pm)} \in H_0^1(B)$ solves the equation $-\Delta \tilde{u}_k^{(\pm)} = (\tilde{f}_k)^{\pm} e^{2\tilde{u}_k}$, then $\pm u_k^{(\pm)} \ge 0$ by the maximum principle. Moreover, in view of the uniform L^1-bound (8), Sobolev embedding theorem yields that

$$\|\tilde{u}_k^{(\pm)}\|_{W^{1,p}(B)} \le C \quad \text{for any } 1 \le p < 2,$$

then a subsequence $\tilde{u}_k^{(\pm)}$ converges weakly in $W^{1,3/2}(B)$ and pointwise almost everywhere. Therefore, from (27) and Theorem 1 in [6], we obtain the uniform bound

$$\|e^{\tilde{u}_k^{(+)}}\|_{L^p(B)} \le C \quad \text{for any } 1 \le p < 4. \tag{29}$$

If we choose $\gamma = 1/10 > 0$, then

$$\gamma \int_M |K_{g_k}| \mathrm{d}\mu_{g_k} \le 4\pi/5$$

for any sufficiently large $k \in \mathbb{N}$. Theorem 1 in [6] can be applied to conclude that

$$\|e^{-\gamma \tilde{u}_k^{(-)}}\|_{L^p(B)} \le C \quad \text{for any } 1 \le p < 4. \tag{30}$$

Thus, by Jensen's inequality and $\tilde{u}_k^{(0)} \le \tilde{u}_k - \tilde{u}_k^{(-)}$, on any disc $D \subset B$, the average of $\gamma u_k^{(0)}$ can be bounded from above by using Hölder's inequality

$$\begin{aligned} \exp\left(\fint_D \gamma \tilde{u}_k^{(0)} \mathrm{d}x \right) &\le \fint_D e^{\gamma \tilde{u}_k^{(0)}} \mathrm{d}x \\ &\le C \left(\int_B e^{2\gamma \tilde{u}_k} \mathrm{d}x \int_B e^{-2\gamma \tilde{u}_k^{(-)}} \mathrm{d}x \right)^{1/2} \\ &\le C \left(\int_M e^{2u_k} \mathrm{d}\mu_{g_b} \right)^{\gamma/2} = C. \end{aligned}$$

By the mean value property of harmonic functions, we obtain the uniform upper bound of $\tilde{u}_k^{(0)} \le C_1$ on $B_{2/3}(x_0)$ for all k. Harnack's inequality applied to $C_1 - \tilde{u}_k^{(0)} \ge 0$ shows that either $|\tilde{u}_k^{(0)}| \le C$ on $B_{1/2}(x_0)$ for all k or $\tilde{u}_k^{(0)} \to -\infty$ uniformly on $B_{1/2}(x_0)$ as $k \to \infty$. In any event, for any ball B upon covering B with finitely many balls $B_{1/2}(x_j), 1 \le j \le J$, and estimating $\tilde{u}_k \le \tilde{u}_k^{(+)} + \tilde{u}_k^{(0)} \le \tilde{u}_k^{(+)} + C_1$ on each ball, we have

$$\|e^{\tilde{u}_k}\|_{L^p(B)} \le C \sum_{1\le j\le J} \|e^{\tilde{u}_k^{(+)}}\|_{L^p(B_{1/2}(x_j))} \le C \quad \text{for any } 1 \le p < 4.$$

Choosing $p = 3$, as before we conclude that $\tilde{u}_k^{(+)} \le C\|u_k^{(+)}\|_{W^{2,3/2}} \le C$ uniformly on any ball B if $k \in \mathbb{N}$ is sufficiently large. In particular, letting $B = B_1(0)$ and bounding $\tilde{u}_k \le \tilde{u}_k^{(+)} + \tilde{u}_k^{(0)} \le C + \tilde{u}_k^{(0)}$ on each $B_{1/2}(x_j)$ in the cover of B, if we suppose that $\tilde{u}_k^{(0)} \to -\infty$ uniformly on some $B_{1/2}(x_{j_0})$, we have also $\tilde{u}_k \to -\infty$ uniformly on $B_{1/2}(x_{j_0})$ and, hence, also on every $B_{1/2}(x_j)$. However, then $\tilde{u}_k \to -\infty$ uniformly also on B, and in view of (10) as $k \to \infty$,

$$\int_{B_{r_k}(x_k)} K_{g_k}^+ \mathrm{d}\mu_{g_k} \le \int_{B_1(0)} \alpha_{\lambda_k}\lambda_k \eta(x_k + r_k x) e^{2\tilde{u}_k}\mathrm{d}x \le C\int_{B_1(0)} e^{2\tilde{u}_k}\mathrm{d}x \to 0, \quad (31)$$

which contradicts to (27). Therefore, $|\tilde{u}_k^{(0)}| \le C$ on $B_{1/2}(0)$ and on every ball $B_{1/2}(x_0)$ for sufficiently large $k \in \mathbb{N}$. By harmonicity, we may assume that a subsequence $\tilde{u}_k^{(0)}$ converges smoothly on every ball $B_{1/2}(x_0)$. Since $\|\tilde{u}_k^{(\pm)}\|_{W^{1,3/2}(B)} \le C$ on every $B = B_1(x_0)$, we conclude that a subsequence $\tilde{u}_k$ converges to $\tilde{u}_\infty$ weakly in $W^{1,3/2}_{loc}$ and almost everywhere on $\mathbb{R}^2$, where $\tilde{u}_\infty \le C$. Fatou's lemma yields that, for any L,

$$\begin{aligned}\int_{B_L(0)} \eta(p_0)e^{2\tilde{u}_\infty}\mathrm{d}x &\le \liminf_{k\to\infty}\int_{B_L(0)} \eta(x_k + r_k x)e^{2\tilde{u}_k}\mathrm{d}x \\ &\le \liminf_{k\to\infty}\int_M \eta e^{2u_k}\mathrm{d}\mu_{g_b} = 1.\end{aligned}$$

Letting $L \to \infty$, we find that $e^{2\tilde{u}_\infty} \in L^1(\mathbb{R}^2)$ with

$$\int_{\mathbb{R}^2} e^{2\tilde{u}_\infty}\mathrm{d}x = \lim_{L\to\infty}\int_{B_L(0)} e^{2\tilde{u}_\infty}\mathrm{d}x \le 1/\eta(p_0). \quad (32)$$

By (16), we may assume $\alpha_{\lambda_k}\lambda_k \to \mu$ as $k \to \infty$ with $\pi \le \mu \le 4\pi$. Recall (28) and set $\tilde{f}_{0k}(x) := \lambda_k^{-1} f_0(x_k + r_k x)$. By Fatou's lemma, we have

$$\begin{aligned}\mu\int_{B_L(0)} \liminf_{k\to\infty}|\tilde{f}_{0k}|e^{2\tilde{u}_\infty}\mathrm{d}x &\le \liminf_{k\to\infty}\int_{B_L(0)} \alpha_{\lambda_k}\lambda_k|\tilde{f}_{0k}|e^{2\tilde{u}_k}\mathrm{d}x \\ &= \liminf_{k\to\infty}\int_{B_L(0)} (\alpha_{\lambda_k}\lambda_k\eta - \tilde{f}_k)e^{2\tilde{u}_k}\mathrm{d}x \\ &\le \liminf_{k\to\infty}\int_M (\alpha_{\lambda_k}\lambda_k\eta - \alpha_{\lambda_k}f_{\lambda_k})e^{2u_k}\mathrm{d}\mu_{g_b} \\ &= \liminf_{k\to\infty}\int_M \alpha_{\lambda_k}\lambda_k\eta e^{2u_k}\mathrm{d}\mu_{g_b} = \mu \le 4\pi.\end{aligned}$$

We now use the condition **UCC** to show that $\tilde{f}_{0k}$ is locally bounded. Indeed, suppose that for some sequence $y_k \to y_0 \in \mathbb{R}^2$ we have $|\tilde{f}_{0k}(y_k)| = |f_0(q_k)|/\lambda_k \to \infty$ as $k \to \infty$, where $q_k = x_k + r_k y_k$, then $q_k \to 0$; hence, $\mathrm{dist}(q_k, M_0) \to 0$ as $k \to \infty$. We assume $\mathrm{dist}(q_k, M_0) < d_0$ for all k. By the condition **UCC**, there exists cones C_{q_k} with vertex q_k such that

$$A_0 \inf_{y\in Q_k}|\tilde{f}_{0k}(y)| = A_0\lambda_k^{-1}\inf_{C_{q_k}}|f_0(q)| \ge \lambda_k^{-1}|f_0(q_k) = |\tilde{f}_{0k}(y_k)| \to \infty \quad (33)$$

as $k \to \infty$, where the cones (up to a rotation)

$$\begin{aligned} Q_k &= \{y | x_k + r_k y \in C_{q_k}\} \\ &= \{y = (y^1, y^2) | 0 < y^2 - y_k^2 < \tan(\theta_0)(y^1 - y_k^1), |y - y_k| < d_0/r_k\} \end{aligned}$$

as $k \to \infty$ suitably exhausts the cone $Q_0 = \{y = (y^1, y^2) | 0 < y^2 - y_0^2 < \tan(\theta_0)(y^1 - y_0^1)\}$. Now, combining

$$\mu \int_{B_L(0) \cap Q_0} e^{2\tilde{u}_\infty} \mathrm{d}x > 0$$

with

$$\mu A_0^{-1} \int_{B_L(0) \cap Q_k} \left(\lim_{k \to \infty} |\tilde{f}_{0k}(y_k)| \right) e^{2\tilde{u}_\infty} \mathrm{d}x \le \mu \int_{B_L(0)} \lim_{k \to \infty} \inf |\tilde{f}_{0k}| e^{2\tilde{u}_\infty} \mathrm{d}x \le C$$

and (33), we obtain the contradiction.

Hence, we may assume that $\tilde{f}_k e^{2\tilde{u}_k} \mathrm{d}x \rightharpoonup \mu(1 + \tilde{f}_{0\infty}) e^{2\tilde{u}_\infty} \mathrm{d}x$ weakly-$*$ in the sense of measure, where $\tilde{f}_{0\infty} \le 0$ is locally bounded from below, then $\tilde{u}_\infty$ weakly solves the equation

$$-\Delta \tilde{u}_\infty = \mu(\eta(p_0) + \tilde{f}_{0\infty}) e^{2\tilde{u}_\infty} \tag{34}$$

with $\int_{\mathbb{R}^2} |\mu(\eta(p_0) + \tilde{f}_{0\infty})| e^{2\tilde{u}_\infty} \mathrm{d}x \le 8\pi$ due to (8). We apply the modified Ding's lemma 2.6 to get

$$\mu \eta(p_0) \int_{\mathbb{R}^2} e^{2\tilde{u}_\infty} \mathrm{d}x \ge 4\pi.$$

Due to $\mu \le 4\pi$ and (32), we have

$$\eta(p_0) \int_{\mathbb{R}^2} e^{2\tilde{u}_\infty} \mathrm{d}x = 1,$$

as well as

$$\mu = 4\pi.$$

Hence, $\tilde{f}_{0\infty} = 0$, and we apply the classification theorem in [7] to obtain

$$\tilde{u}_\infty(x) = \log \left(\frac{4}{4 + 4\pi\eta(p_0)|x - x_\infty|^2} \right).$$

We can find $z_k \to x_\infty$ such that $\tilde{u}_k(z_k) = \sup_{\mathbb{R}^2} \tilde{u}_k$, then we may choose $p_k^{(i)} = x_k + r_k z_k$ and a new Euclidean coordinates x around $p_k^{(i)} = 0$ such that

$$\tilde{u}_k \to \log \left(\frac{4}{4 + 4\pi\eta(p_0)|x|^2} \right)$$

in $H^2_{loc}(\mathbb{R}^2)$. Since the weighted volumes of all bubbles obtained by rescaling the normalized metrics can add up to at most one, i.e.

$$\sum_{1 \le i \le i_0} \int_{\mathbb{R}^2} \eta(p^i_\infty) e^{2\tilde{u}^i_\infty} \le \int_M \eta e^{2u_\lambda} = 1,$$

$i_0 = 1$, and we complete our proof. □

Acknowledgments. The authors would like to thank the referees for their very valuable suggestions.

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Received for publication July 2023;; early access December 2023.